

# Energy decay for Maxwell's equations with Ohm's law on partially cubic domains \*

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**Abstract .-** We prove a polynomial energy decay for the Maxwell's equations with Ohm's law on partially cubic domains with trapped rays.

**Keywords .-** Maxwell's equation; decay estimates; trapped ray.

## 1 Introduction

The problems dealing with Maxwell's equations with nonzero conductivity are not only theoretical interesting but also very important in many industrial applications (see e.g. [3], [7], [8]).

Let  $\Omega$  be a bounded open connected region in  $\mathbb{R}^3$ , with a smooth boundary  $\partial\Omega$ . We suppose that  $\Omega$  is simply connected and  $\partial\Omega$  has only one connected component. The domain  $\Omega$  is occupied by an electromagnetic medium of constant electric permittivity  $\varepsilon_o$  and constant magnetic permeability  $\mu_o$ . Let  $E$  and  $H$  denote the electric and magnetic fields respectively. The Maxwell's equations with Ohm's law are described by

$$\left\{ \begin{array}{ll} \varepsilon_o \partial_t E - \text{curl } H + \sigma E = 0 & \text{in } \Omega \times [0, +\infty) \\ \mu_o \partial_t H + \text{curl } E = 0 & \text{in } \Omega \times [0, +\infty) \\ \text{div}(\mu_o H) = 0 & \text{in } \Omega \times [0, +\infty) \\ E \times \nu = H \cdot \nu = 0 & \text{on } \partial\Omega \times [0, +\infty) \\ (E, H)(\cdot, 0) = (E_o, H_o) & \text{in } \Omega . \end{array} \right. \quad (1.1)$$

Here,  $(E_o, H_o)$  are the initial data in the energy space  $L^2(\Omega)^6$  and  $\nu$  denotes the outward unit normal vector to  $\partial\Omega$ . The conductivity is such that  $\sigma \in L^\infty(\Omega)$  and  $\sigma \geq 0$ . It is well-known that when the conductivity is identically null, then the above system is conservative and when  $\sigma$  is bounded from below by a positive constant, then an exponential energy decay rate holds for the Maxwell's equations with Ohm's law in the energy space. The situation becomes more delicate when we only assume that

$$\sigma(x) \geq \text{constant} > 0 \quad \forall x \in \omega$$

for some non-empty connected open subset  $\omega$  of  $\Omega$ . Observe that the condition  $\text{div}(\varepsilon_o E) = 0$  in  $\Omega \times [0, +\infty)$  does not appear because the free divergence is not preserved by the Maxwell's equations with Ohm's law. Here we know that the above system is dissipative and its energy tends to zero in large time. However, we would like to establish the energy decay rate as well. In the field of control

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theory, the exponential energy decay rate of a linear dissipative system is deduced from an observability estimate. Precisely, in order to get an exponential decay rate in the energy space we should have the following observability inequality

$$\exists C, T_c > 0 \quad \forall \zeta \geq 0 \quad \int_{\Omega} |(E, H)(\cdot, \zeta)|^2 dx \leq C \int_{\zeta}^{\zeta+T_c} \int_{\Omega} \sigma |E|^2 dx dt$$

or simply, in virtue of a semigroup property,

$$\exists C, T_c > 0 \quad \int_{\Omega} |(E_o, H_o)|^2 dx \leq C \int_0^{T_c} \int_{\Omega} \sigma |E|^2 dx dt$$

for any initial data  $(E_o, H_o)$  in the energy space  $L^2(\Omega)^6$ . We can also look for establishing the above observability inequality for any initial data in the energy space intersecting suitable invariant subspaces but not with the condition  $\operatorname{div} E_o = 0$  in  $\Omega$ . Such estimate is established in [11] under the geometric control condition of Bardos, Lebeau and Rauch [2] for the scalar wave operator and when the conductivity has the property that  $\sigma(x) \geq \text{constant} > 0$  for all  $x \in \omega$  and  $\sigma(x) = 0$  for all  $x \in \Omega \setminus \overline{\omega}$ . From now, we consider a subset  $\omega$  such that the geometric control condition for the scalar wave operator or other assumptions based on the multiplier method fail. In such geometry, we do not hope an exponential energy decay rate in the energy space. Our geometry (described precisely in Section 3) presents parallel trapped rays and can be compared to the one in [12] or in [4],[10] for the two dimensional case. It generalises the cube (see [8]) and therefore explicit and analytical results are harder to obtain. Our main result gives a polynomial energy decay with regular initial data. Our proof is based on a new kind of observation inequality (see (4.33) below) which can also be seen as an interpolation estimate. It relies with the construction of a particular solution for the operator  $i\partial_s + h(\Delta - \partial_t^2)$  inspired by the gaussian beam techniques. Also the dispersion property for the one dimensional Schrödinger operator will play a key role.

The plan of the paper is as follows. In the next section, we recall the known results about the Maxwell's equations with Ohm's law that will be used in the following. Section 3 contains the statement of our main result, while Section 4 is concerned with its proof. In Section 5, we present the interpolation estimate, while Section 6 includes its proof. Finally, two appendix are added dealing with inequalities involving Fourier analysis.

## 2 The Maxwell's equations with Ohm's law

We begin to recall some well-known results concerning the Maxwell's equations with Ohm's law: well-posedness, energy identity, standard orthogonal decomposition and asymptotic behaviour in time of the energy of the electromagnetic field.

### 2.1 Well-posedness of the problem

Let us introduce the spaces

$$\mathcal{V} = L^2(\Omega)^3 \times \left\{ G \in L^2(\Omega)^3; \operatorname{div} G = 0, G \cdot \nu_{|\partial\Omega} = 0 \right\}, \quad (2.1.1)$$

$$\mathcal{W} = \left\{ (F, G) \in L^2(\Omega)^6; \operatorname{curl} F \in L^2(\Omega)^3, F \times \nu_{|\partial\Omega} = 0, \right. \\ \left. \operatorname{div} G = 0, G \cdot \nu_{|\partial\Omega} = 0, \operatorname{curl} G \in L^2(\Omega)^3 \right\}. \quad (2.1.2)$$

It is well-known that if  $(E_o, H_o) \in \mathcal{V}$ , there is a unique weak solution  $(E, H) \in C^0([0, +\infty), \mathcal{V})$ . Further, if  $(E_o, H_o) \in \mathcal{W}$ , there is a unique strong solution  $(E, H) \in C^0([0, +\infty), \mathcal{W}) \cap C^1([0, +\infty), \mathcal{V})$ . Let us define the functionals of energy

$$\mathcal{E}(t) = \frac{1}{2} \int_{\Omega} \left( \varepsilon_o |E(x, t)|^2 + \mu_o |H(x, t)|^2 \right) dx, \quad (2.1.3)$$

$$\mathcal{E}_1(t) = \frac{1}{2} \int_{\Omega} \left( \varepsilon_o |\partial_t E(x, t)|^2 + \mu_o |\partial_t H(x, t)|^2 \right) dx. \quad (2.1.4)$$

We can easily check that the energy  $\mathcal{E}$  is a continuous positive non-increasing real function on  $[0, +\infty)$  and further for any initial data  $(E_o, H_o) \in \mathcal{W}$ ,

$$\frac{d}{dt} \mathcal{E}(t) + \int_{\Omega} \sigma(x) |E(x, t)|^2 dx = 0, \quad (2.1.5)$$

and for any  $t_2 > t_1 \geq 0$ ,

$$\mathcal{E}(t_2) - \mathcal{E}(t_1) + \int_{t_1}^{t_2} \int_{\Omega} \sigma(x) |E(x, t)|^2 dx dt = 0, \quad (2.1.6)$$

$$\mathcal{E}_1(t_2) - \mathcal{E}_1(t_1) + \int_{t_1}^{t_2} \int_{\Omega} \sigma(x) |\partial_t E(x, t)|^2 dx dt = 0. \quad (2.1.7)$$

## 2.2 Orthogonal decomposition

Both  $E$  and  $\mu_o H$  can be described, by means of the scalar and vector potentials  $p$  and  $A$  with the Coulomb gauge, in a unique way as follows.

**Proposition 2.1** -. *For any initial data  $(E_o, H_o) \in \mathcal{W}$ , there is a unique  $(p, A) \in C^1([0, +\infty), H_0^1(\Omega)) \times C^2([0, +\infty), H^1(\Omega)^3)$  such that  $(E, H)$  the solution of (1.1) the Maxwell's equations with Ohm's law satisfies*

$$\begin{cases} E = -\nabla p - \partial_t A \\ \mu_o H = \text{curl } A \end{cases} \quad (2.2.1)$$

$$\begin{cases} \varepsilon_o \mu_o \partial_t^2 A + \text{curl curl } A = \mu_o (-\varepsilon_o \partial_t \nabla p + \sigma E) & \text{in } \Omega \times [0, +\infty) \\ \text{div } A = 0 & \text{in } \Omega \times [0, +\infty) \\ A \times \nu = 0 & \text{on } \partial\Omega \times [0, +\infty) \end{cases} \quad (2.2.2)$$

and we have the following relations

$$\|E\|_{L^2(\Omega)^3}^2 = \|\nabla p\|_{L^2(\Omega)^3}^2 + \|\partial_t A\|_{L^2(\Omega)^3}^2, \quad (2.2.3)$$

$$\|\varepsilon_o \partial_t \nabla p\|_{L^2(\Omega)^3} \leq \|\sigma E\|_{L^2(\Omega)^3}, \quad (2.2.4)$$

$$\exists c > 0 \quad \|A\|_{L^2(\Omega)^3}^2 \leq c \|\text{curl } A\|_{L^2(\Omega)^3}^2. \quad (2.2.5)$$

Further, since  $\text{curl } H \in L^2(\Omega)^3$ ,  $\text{curl curl } A \in L^2(\Omega)^3$  and  $\text{div } A \in H_0^1(\Omega)$ .

The proof is essentially given in [11, page 121] from a Hodge decomposition and is omitted here. Now, the vector field  $A$  has the nice property of free divergence and satisfies a second order vector wave equation with homogeneous boundary condition  $A \times \nu = \text{div } A = 0$  and with a second member in  $C^1([0, +\infty), L^2(\Omega)^3)$  bounded by  $2\mu_o \|\sigma E\|_{L^2(\Omega)^3}$ . For the sake of simplicity, we assume from now that  $\varepsilon_o \mu_o = 1$ .

### 2.3 Invariant subspaces, asymptotic behavior and exponential energy decay

Let  $\omega_+$  be a non-empty connected open subset of  $\Omega$  with a Lipschitz boundary  $\partial\omega_+$ . Suppose that  $\sigma \in L^\infty(\Omega)$  with the property that  $\sigma(x) \geq \text{constant} > 0$  for all  $x \in \omega_+$  and  $\sigma(x) = 0$  for all  $x \in \Omega \setminus \overline{\omega_+}$ . Define  $\omega_- = \Omega \setminus \text{supp}\sigma$  and suppose that its boundary  $\partial\omega_-$  is Lipschitz and has no more than two connected components  $\gamma_1, \gamma_2$ .

We recall that the range of the curl,  $\text{curl } H^1(\omega_-)^3$ , is closed in  $L^2(\omega_-)^3$  (see [6, page 257] or [5, page 54]) and

$$\text{curl } H^1(\omega_-)^3 = \left\{ U \in L^2(\omega_-)^3; \text{div } U = 0 \text{ in } \omega_-, \int_{\gamma_i} U \cdot \nu = 0 \text{ for } i \in \{1, 2\} \right\}. \quad (2.3.1)$$

Its orthogonal space for the  $(L^2(\omega_-))^3$  norm is

$$\left( \text{curl } H^1(\omega_-)^3 \right)^\perp = \left\{ V \in L^2(\omega_-)^3; \text{curl } V = 0 \text{ in } \omega_-, V \times \nu = 0 \text{ on } \partial\omega_- \right\}. \quad (2.3.2)$$

Let us introduce  $\mathcal{S}_\sigma = \left( \text{curl } H^1(\omega_-)^3 \cap L^2(\Omega)^3 \right) \times L^2(\Omega)^3$ . The space  $\mathcal{W} \cap \mathcal{S}_\sigma$  is stable for the system of Maxwell's equations with Ohm's law, which can be seen by multiplying by  $V \in \left( \text{curl } H^1(\omega_-)^3 \right)^\perp$  the equation  $\varepsilon_o \partial_t E - \text{curl } H + \sigma E = 0$ . Then, we can add the following well-posedness result. If  $(E_o, H_o) \in \mathcal{W} \cap \mathcal{S}_\sigma$ , there is a unique solution  $(E, H) \in C^0([0, +\infty), \mathcal{W} \cap \mathcal{S}_\sigma) \cap C^1([0, +\infty), \mathcal{V} \cap \mathcal{S}_\sigma)$ .

It has been proved (see [11, page 124]) that if  $\omega_-$  is a non-empty connected open set then  $\lim_{t \rightarrow +\infty} \mathcal{E}(t) = 0$  for any initial data  $(E_o, H_o) \in \mathcal{V} \cap \mathcal{S}_\sigma$ . Further, the following result (see [11, page 124]) plays a key role.

**Proposition 2.2** -. *If  $\omega_-$  is a non-empty connected open set, then there exists  $c > 0$  such that for all initial data  $(E_o, H_o) \in \mathcal{W} \cap \mathcal{S}_\sigma$  of the system (1.1) of Maxwell's equations with Ohm's law, we have*

$$\forall t \geq 0 \quad \mathcal{E}(t) \leq c \mathcal{E}_1(t). \quad (2.3.3)$$

**Remark 2.3** -. The estimate (2.3.3) is still true if  $\partial\omega_+ \cap \partial\Omega \neq \emptyset$ . Indeed, the proof given in [11, page 127] can be divided into two steps. In the first step, we begin to establish the existence of  $c > 0$  such that

$$\|\nabla p\|_{L^2(\omega_+)^3}^2 + \|\partial_t A\|_{L^2(\Omega)^3}^2 + \|H\|_{L^2(\Omega)^3}^2 \leq c \left( \mathcal{E}_1(t) + \sqrt{\mathcal{E}(t)} \sqrt{\mathcal{E}_1(t)} \right) \quad (2.3.4)$$

for any  $(E_o, H_o) \in \mathcal{W} \cap \mathcal{S}_\sigma$ . Here, we used a standard compactness-uniqueness argument for  $H$ , (2.2.5) of Proposition 2.1 for  $\partial_t A$ , and for  $\nabla p$  from the fact that  $\sigma(x) \geq \text{constant} > 0$  for all  $x \in \omega_+$  and (2.1.5). Till now, we did not need that  $\omega_-$  is a connected set. The second step (see [11, page 128]) did consist to prove that

$$\|\nabla p\|_{L^2(\omega_-)^3}^2 \leq c \left( \|\nabla p\|_{L^2(\omega_+)^3}^2 + \|\partial_t A\|_{L^2(\Omega)^3}^2 \right). \quad (2.3.5)$$

Finally, we concluded by virtue of (2.2.3) of Proposition 2.1. This last estimate becomes easier to obtain under the assumption  $\partial\omega_+ \cap \partial\Omega \neq \emptyset$  and without adding the hypothesis saying that  $\omega_-$  is a connected set. Indeed, since  $(E_o, H_o) \in \mathcal{W} \cap \mathcal{S}_\sigma$  and  $-\Delta p = \text{div } E$ ,  $p \in H_0^1(\Omega)$  solves the following elliptic system

$$\begin{cases} \Delta p = 0 & \text{in } \omega_- \\ p \in H^{1/2}(\partial\omega_-) \\ p = 0 & \text{on } \partial\omega_+ \cap \partial\Omega \neq \emptyset. \end{cases} \quad (2.3.6)$$

Thus, by the elliptic regularity, the trace theorem and the Poincaré inequality, we have the following estimate

$$\|\nabla p\|_{L^2(\omega_-)^3} \leq c_1 \|p\|_{H^{1/2}(\partial\omega_+)} \leq c_2 \|p\|_{H^{1/2}(\partial\omega_+)} \leq c_3 \|\nabla p\|_{L^2(\omega_+)^3} \quad (2.3.7)$$

for suitable constants  $c_1, c_2, c_3 > 0$ . Hence, combining (2.3.4) and (2.3.7) with (2.2.3), (2.3.3) follows if  $\partial\omega_+ \cap \partial\Omega \neq \emptyset$ .

The exponential energy decay rate for the Maxwell's equations with Ohm's law in the energy space is as follows.

**Proposition 2.4 -** *Let  $\vartheta$  be a subset of  $\Omega$  such that any generalized ray of the scalar wave operator  $\partial_t^2 - \Delta$  meets  $\bar{\vartheta}$ . Suppose that  $\bar{\vartheta} \cap \Omega \subset \omega_+$ . Further if  $\partial\omega_+ \cap \partial\Omega \neq \emptyset$  or  $\omega_-$  is a non-empty connected open set, then there exist  $c > 0$  and  $\beta > 0$  such that for all initial data  $(E_o, H_o) \in \mathcal{V} \cap \mathcal{S}_\sigma$  of the system (1.1) of Maxwell's equations with Ohm's law, we have*

$$\forall t \geq 0 \quad \mathcal{E}(t) \leq ce^{-\beta t} \mathcal{E}(0) . \quad (2.3.8)$$

The proof of Proposition 2.4 is done in [11, page 129] when  $\omega_-$  is a non-empty connected open set. Here, we simply recall the key points of the proof. From the geometric control condition, the following estimate holds without using the fact that  $\omega_-$  is a non-empty connected open set.

$$\exists C, T_c > 0 \quad \forall \zeta \geq 0 \quad \mathcal{E}_1(\zeta) \leq C \int_\zeta^{T_c+\zeta} \int_\Omega \left( \sigma |\partial_t E|^2 + \sigma |E|^2 \right) dx dt \quad (2.3.9)$$

for any initial data  $(E_o, H_o) \in \mathcal{W} \cap \mathcal{S}_\sigma$ . Next by (2.3.3), we deduced that

$$\exists C, T_c > 0 \quad \forall \zeta \geq 0 \quad \mathcal{E}(\zeta) + \mathcal{E}_1(\zeta) \leq C \int_\zeta^{T_c+\zeta} \int_\Omega \left( \sigma |\partial_t E|^2 + \sigma |E|^2 \right) dx dt . \quad (2.3.10)$$

Finally, we concluded by virtue of a semigroup property. The proof works as well when  $\partial\omega_+ \cap \partial\Omega \neq \emptyset$  thanks to Remark 2.3.

### 3 Geometric setting and main result

Let us introduce the geometry on which we work in this paper.

We set  $D(r_1, r_2) = \{(x_1, x_2) \in \mathbb{R}^2; |x_1| < r_1, |x_2| < r_2\}$  where  $r_1, r_2 > 0$ . Let  $m_1, m_2, \rho > 0$ . We choose  $\Omega$  a connected open set in  $\mathbb{R}^3$  bounded by  $\Gamma_1, \Gamma_2, \Upsilon$  where

$$\Gamma_1 = \overline{D(m_1, m_2)} \times \{\rho\}, \text{ with boundary } \partial\Gamma_1,$$

$$\Gamma_2 = \overline{D(m_1, m_2)} \times \{-\rho\}, \text{ with boundary } \partial\Gamma_2,$$

$$\Upsilon \text{ is a surface with boundary } \partial\Upsilon = \partial\Gamma_1 \cup \partial\Gamma_2.$$

Therefore, the boundary of  $\Omega$  is  $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Upsilon$ . Further, we suppose that  $\partial\Omega$  is  $C^\infty$  with  $\Upsilon \subset (\mathbb{R}^2 \setminus D(m_1, m_2)) \times \mathbb{R}$ . In particular,  $\Omega$  is simply connected and  $\partial\Omega$  has only one connected component.

Let  $\Theta$  be a small neighborhood of  $\Upsilon$  in  $\mathbb{R}^3$  such that  $\Theta \cap D(M_1, M_2) \times [-\rho, \rho] = \emptyset$  for some  $M_1 \in (0, m_1)$  and  $M_2 \in (0, m_2)$ . Further, we suppose that the boundaries  $\partial(\Theta \cap \Omega)$  and  $\partial(\Omega \setminus \overline{\Theta \cap \Omega})$  are at least Lipschitz.

After these preparations, we are now able to state our main result.

**Theorem 3.1 -** *Let  $\omega = \Theta \cap \Omega$ . If  $\sigma \in L^\infty(\Omega)$  is such that  $\sigma(x) \geq \text{constant} > 0$  for all  $x \in \omega$  and  $\sigma(x) = 0$  for all  $x \in \Omega \setminus \bar{\omega}$ , then there exist  $c > 0$  and  $\gamma > 0$  such that for any  $t \geq 0$*

$$\mathcal{E}(t) \leq \frac{c}{t^\gamma} (\mathcal{E}(0) + \mathcal{E}_1(0)) \quad (3.1)$$

for every solution of the system (1.1) of Maxwell's equations with Ohm's law with initial data  $(E_o, H_o)$  in  $\mathcal{W} \cap \mathcal{S}_\sigma$ .

**Remark 3.2** -. From now  $\omega = \Theta \cap \Omega$  and  $\sigma \in L^\infty(\Omega)$  is such that  $\sigma(x) \geq \text{constant} > 0$  for all  $x \in \omega$  and  $\sigma(x) = 0$  for all  $x \in \Omega \setminus \bar{\omega}$ . Notice that  $\omega$  and  $\Omega \setminus \bar{\omega}$  are two non-empty connected open sets with Lipschitz boundaries. Therefore by Proposition 2.2, there exists  $c > 0$  such that for all initial data  $(E_o, H_o) \in \mathcal{W} \cap \mathcal{S}_\sigma$  of the system (1.1) of Maxwell's equations with Ohm's law,  $\mathcal{E}(t) \leq c \mathcal{E}_1(t)$  for any  $t \geq 0$ .

**Remark 3.3** -. Notice the existence of trapped rays bouncing up and down from  $\Gamma_1$  to  $\Gamma_2$ .

## 4 Proof of the main result

Let us consider the solution  $U$  of the following system

$$\left\{ \begin{array}{ll} \partial_t^2 U + \text{curl curl } U = 0 & \text{in } \Omega \times \mathbb{R} \\ \text{div } U = 0 & \text{in } \Omega \times \mathbb{R} \\ U \times \nu = 0 & \text{on } \partial\Omega \times \mathbb{R} \\ (U(\cdot, 0), \partial_t U(\cdot, 0)) = (U^0, U^1) & \text{in } \Omega, \\ (U^0, U^1) \in \mathcal{X} & , \\ (U^1, \text{curl curl } U^0) \in \mathcal{X} & , \end{array} \right. \quad (4.1)$$

where

$$\mathcal{X} = \left\{ (F, G) \in L^2(\Omega)^6; \text{ curl } F \in L^2(\Omega)^3, F \times \nu|_{\partial\Omega} = 0, \text{div } F = 0, \text{div } G = 0 \right\}. \quad (4.2)$$

It is well-known that the above system is well-posed with a unique solution  $U$  such that  $(U(\cdot, t), \partial_t U(\cdot, t))$  and  $(\partial_t U(\cdot, t), \partial_t^2 U(\cdot, t))$  belong to  $\mathcal{X}$  for any  $t \in \mathbb{R}$ . Let us define the following two conservations of energies.

$$\mathcal{G}(U, 0) = \mathcal{G}(U, t) \equiv \int_{\Omega} \left( |\partial_t U(x, t)|^2 + |\text{curl } U(x, t)|^2 \right) dx, \quad (4.3)$$

$$\mathcal{G}(\partial_t U, 0) = \mathcal{G}(\partial_t U, t) \equiv \int_{\Omega} \left( |\text{curl curl } U(x, t)|^2 + |\text{curl } \partial_t U(x, t)|^2 \right) dx. \quad (4.4)$$

Further, for such solution  $U$ , the following two inequalities hold by standard compactness-uniqueness argument and classical embedding (see [1] and [5, page 50]).

$$\mathcal{G}(U, t) \leq c \mathcal{G}(\partial_t U, t), \quad (4.5)$$

$$\|U(\cdot, t)\|_{H^1(\Omega)^3}^2 \leq c \|\text{curl } U(\cdot, t)\|_{L^2(\Omega)^3}^2, \quad (4.6)$$

for some  $c > 0$  and any  $t \in \mathbb{R}$ .

Since  $\Theta$  is a small neighborhood of  $\Upsilon$  in  $\mathbb{R}^3$  such that  $\Theta \cap D(M_1, M_2) \times [-\rho, \rho] = \emptyset$  for some  $M_1 \in (0, m_1)$  and  $M_2 \in (0, m_2)$ , there exists a positive real number  $r_o < \min(m_1 - M_1, m_2 - M_2, \rho)/2$  such that  $D(m_1, m_2) \setminus D(m_1 - 2r_o, m_2 - 2r_o) \times (\rho - 2r_o, \rho + 2r_o) \cup (-\rho - 2r_o, -\rho + 2r_o) \subset \Theta$ . Now, we define

$$\omega_o = D(m_1 - r_o, m_2 - r_o) \times (\rho - 2r_o, \rho - r_o). \quad (4.7)$$

**Proposition 4.1** -. There exist  $h_o, c, \gamma > 0$  such that for any  $T_o > 0$ ,  $\zeta \geq 0$  and  $h \in (0, h_o]$ , the solution  $U$  of (4.1) satisfies

$$\int_{\zeta + c \frac{1}{h^\gamma}}^{T_o + \zeta + c \frac{1}{h^\gamma}} \int_{\omega_o} |\partial_t U|^2 dx dt \leq c \frac{1}{h^\gamma} \int_{\zeta}^{\zeta + 2c \frac{1}{h^\gamma}} \int_{\omega} \left( |\partial_t U|^2 + |U|^2 \right) dx dt + ch \mathcal{G}(\partial_t U, \zeta). \quad (4.8)$$

We shall leave the proof of Proposition 4.1 till later (see Section 5). Now we turn to prove Theorem 3.1.

We start by choosing  $\widetilde{\omega}_o \subset \omega_o$  such that  $\omega \cup \widetilde{\omega}_o$  is a non-empty connected open set and such that the boundaries  $\partial(\omega \cup \widetilde{\omega}_o)$  and  $\partial(\Omega \setminus \overline{\omega \cup \widetilde{\omega}_o})$  are Lipschitz. Notice that  $\partial(\omega \cup \widetilde{\omega}_o) \cap \partial\Omega \neq \emptyset$  and there exists  $\vartheta$  a subset of  $\Omega$  such that  $\overline{\vartheta} \cap \Omega \subset (\omega \cup \widetilde{\omega}_o)$  and such that any generalized ray of the scalar wave operator  $\partial_t^2 - \Delta$  meets  $\overline{\vartheta}$ .

Let  $\zeta, T_h \geq 0$ . Let  $(\widetilde{E}, \widetilde{H})$  denote the electromagnetic field of the following Maxwell's equations with Ohm's law

$$\left\{ \begin{array}{ll} \varepsilon_o \partial_t \widetilde{E} - \operatorname{curl} \widetilde{H} + (\sigma + 1_{|\widetilde{\omega}_o}) \widetilde{E} = 0 & \text{in } \Omega \times [0, +\infty) \\ \mu_o \partial_t \widetilde{H} + \operatorname{curl} \widetilde{E} = 0 & \text{in } \Omega \times [0, +\infty) \\ \operatorname{div}(\mu_o \widetilde{H}) = 0 & \text{in } \Omega \times [0, +\infty) \\ \widetilde{E} \times \nu = \widetilde{H} \cdot \nu = 0 & \text{on } \partial\Omega \times [0, +\infty) \\ (\widetilde{E}, \widetilde{H})(\cdot, \zeta + T_h) = (E, H)(\cdot, \zeta + T_h) & \text{in } \Omega. \end{array} \right. \quad (4.9)$$

The conductivity  $(\sigma + 1_{|\widetilde{\omega}_o})$  is such that  $(\sigma + 1_{|\widetilde{\omega}_o}) \geq \text{constant} > 0$  in  $\omega \cup \widetilde{\omega}_o$  and  $(\sigma + 1_{|\widetilde{\omega}_o}) = 0$  in  $\Omega \setminus \overline{(\omega \cup \widetilde{\omega}_o)}$ . Also notice that  $(E, H) \in \mathcal{W} \cap \mathcal{S}_\sigma \subset (\mathcal{V} \cap \mathcal{S}_{\sigma+1_{|\widetilde{\omega}_o}})$ . Therefore by Proposition 2.4, there exist  $c, \beta > 0$  (independent of  $\zeta, T_h$ ) such that for any  $t \geq 0$  we have

$$\int_{\Omega} \left( \varepsilon_o \left| \widetilde{E}(x, t + \zeta + T_h) \right|^2 + \mu_o \left| \widetilde{H}(x, t + \zeta + T_h) \right|^2 \right) dx \leq ce^{-\beta t} \mathcal{E}(\zeta + T_h). \quad (4.10)$$

On the other hand, let  $(\overline{E}, \overline{H}) = (\widetilde{E}, \widetilde{H}) - (E, H)$ . Then it solves

$$\left\{ \begin{array}{ll} \varepsilon_o \partial_t \overline{E} - \operatorname{curl} \overline{H} + (\sigma + 1_{|\widetilde{\omega}_o}) \overline{E} = -1_{|\widetilde{\omega}_o} E & \text{in } \Omega \times [0, +\infty) \\ \mu_o \partial_t \overline{H} + \operatorname{curl} \overline{E} = 0 & \text{in } \Omega \times [0, +\infty) \\ \operatorname{div}(\mu_o \overline{H}) = 0 & \text{in } \Omega \times [0, +\infty) \\ \overline{E} \times \nu = \overline{H} \cdot \nu = 0 & \text{on } \partial\Omega \times [0, +\infty) \\ (\overline{E}(\cdot, \zeta + T_h), \overline{H}(\cdot, \zeta + T_h)) = (0, 0) & \text{in } \Omega, \end{array} \right. \quad (4.11)$$

and by a standard energy method and the fact that  $\widetilde{\omega}_o \subset \omega_o$ , we get that for any  $t \geq 0$

$$\int_{\Omega} \left( \varepsilon_o \left| \overline{E}(x, t + \zeta + T_h) \right|^2 + \mu_o \left| \overline{H}(x, t + \zeta + T_h) \right|^2 \right) dx \leq \frac{t}{\varepsilon_o} \int_{\zeta+T_h}^{t+\zeta+T_h} \int_{\omega_o} |E(x, s)|^2 dx ds. \quad (4.12)$$

Now we are able to bound the quantity  $\mathcal{E}(\zeta + T_h) = \mathcal{E}(t + \zeta + T_h) + \int_{\zeta+T_h}^{t+\zeta+T_h} \int_{\Omega} \sigma(x) |E(x, s)|^2 dx ds$  as follows. By using (4.10) and (4.12), we deduce that

$$\mathcal{E}(\zeta + T_h) \leq 2ce^{-\beta t} \mathcal{E}(\zeta + T_h) + \frac{2t}{\varepsilon_o} \int_{\zeta+T_h}^{t+\zeta+T_h} \int_{\omega_o} |E(x, s)|^2 dx ds + \int_{\zeta+T_h}^{t+\zeta+T_h} \int_{\Omega} \sigma(x) |E(x, s)|^2 dx ds \quad (4.13)$$

which implies by taking  $t$  large enough, the existence of constants  $C, T_c > 1$  such that

$$\mathcal{E}(\zeta + T_h) \leq C \int_{\zeta+T_h}^{T_c+\zeta+T_h} \left( \int_{\Omega} \sigma |E|^2 + \int_{\omega_o} |E|^2 dx \right) dx dt. \quad (4.14)$$

Recall the existence of the vector potential  $A$  from Proposition 2.1 and let  $U$  be the solution of

$$\left\{ \begin{array}{ll} \partial_t^2 U + \operatorname{curl} \operatorname{curl} U = 0 & \text{in } \Omega \times \mathbb{R} \\ \operatorname{div} U = 0 & \text{in } \Omega \times \mathbb{R} \\ U \times \nu = 0 & \text{on } \partial\Omega \times \mathbb{R} \\ (U, \partial_t U)(\cdot, \zeta) = (A, \partial_t A)(\cdot, \zeta) & \text{in } \Omega, \end{array} \right. \quad (4.15)$$

then by a standard energy method, for any  $T_1 > 0$

$$\int_{\zeta}^{\zeta+T_1} \int_{\Omega} \left( |\partial_t (U - A)|^2 + |\operatorname{curl} (U - A)|^2 \right) dx dt \leq T_1^2 \int_{\zeta}^{\zeta+T_1} \int_{\Omega} \mu_o |-\varepsilon_o \partial_t \nabla p + \sigma E|^2 dx dt \quad (4.16)$$

which implies from (2.2.4) of Proposition 2.1 that

$$\int_{\zeta}^{\zeta+T_1} \int_{\Omega} \left( |\partial_t (U - A)|^2 + |\operatorname{curl} (U - A)|^2 \right) dx dt \leq 4\mu_o T_1^2 \int_{\zeta}^{\zeta+T_1} \int_{\Omega} |\sigma E|^2 dx dt . \quad (4.17)$$

Now we are able to bound the quantity  $\mathcal{E}(\zeta) = \mathcal{E}(\zeta + T_h) + \int_{\zeta}^{\zeta+T_h} \int_{\Omega} \sigma(x) |E(x, s)|^2 dx ds$  as follows. Since  $E = -\nabla p + \partial_t (U - A) - \partial_t U$ , we deduce by (4.14) and (4.17) that

$$\begin{aligned} \mathcal{E}(\zeta) &\leq C \int_{\zeta+T_h}^{T_c+\zeta+T_h} \left( \int_{\Omega} \sigma |E|^2 dx dt + \int_{\omega_o} |-\nabla p + \partial_t (U - A) - \partial_t U|^2 dx \right) dt + \int_{\zeta}^{\zeta+T_h} \int_{\Omega} \sigma |E|^2 dx dt \\ &\leq C(1 + T_h^2) \int_{\zeta}^{T_c+\zeta+T_h} \int_{\Omega} \left( \sigma |E|^2 + |\nabla p|^2 \right) dx dt + C \int_{\zeta+T_h}^{T_c+\zeta+T_h} \int_{\omega_o} |\partial_t U|^2 dx dt . \end{aligned} \quad (4.18)$$

Here and hereafter,  $C$  will be used to denote a generic constant, not necessarily the same in any two places.

Now we fix  $T_h = c \frac{1}{h^\gamma}$  where  $c$  and  $\gamma$  are given by Proposition 4.1. Taking  $T_o = T_c$  in Proposition 4.1, we obtain that for any  $\zeta \geq 0$  and  $h \in (0, h_o]$ ,

$$\int_{\zeta+T_h}^{T_c+\zeta+T_h} \int_{\omega_o} |\partial_t U|^2 dx dt \leq CT_h \int_{\zeta}^{\zeta+2T_h} \int_{\omega} \left( |\partial_t U|^2 + |U|^2 \right) dx dt + ch \mathcal{G}(\partial_t U, \zeta) . \quad (4.19)$$

But

$$\begin{aligned} \int_{\zeta}^{\zeta+2T_h} \int_{\omega} |\partial_t U|^2 dx dt &= \int_{\zeta}^{\zeta+2T_h} \int_{\omega} | -E - \nabla p + \partial_t (U - A) |^2 dx dt \\ &\leq CT_h^2 \int_{\zeta}^{\zeta+2T_h} \int_{\Omega} \left( \sigma |E|^2 + |\nabla p|^2 \right) dx dt \end{aligned} \quad (4.20)$$

and

$$\begin{aligned} \int_{\zeta}^{\zeta+2T_h} \int_{\omega} |U|^2 dx dt &\leq 2 \int_{\zeta}^{\zeta+2T_h} \int_{\Omega} |U - A|^2 dx dt + 2 \int_{\zeta}^{\zeta+2T_h} \int_{\omega} |A|^2 dx dt \\ &\leq C \int_{\zeta}^{\zeta+2T_h} \int_{\Omega} |\operatorname{curl} (U - A)|^2 dx dt + 2 \int_{\zeta}^{\zeta+2T_h} \int_{\omega} |A|^2 dx dt \\ &\leq CT_h^2 \int_{\zeta}^{\zeta+2T_h} \int_{\Omega} \sigma |E|^2 dx dt + 2 \int_{\zeta}^{\zeta+2T_h} \int_{\omega} |A|^2 dx dt \end{aligned} \quad (4.21)$$

therefore (4.19) becomes

$$\int_{\zeta+T_h}^{T_c+\zeta+T_h} \int_{\omega_o} |\partial_t U|^2 dx dt \leq CT_h^3 \int_{\zeta}^{\zeta+2T_h} \left( \int_{\Omega} \left( \sigma |E|^2 + |\nabla p|^2 \right) dx + \int_{\omega} |A|^2 dx \right) dt + ch \mathcal{G}(\partial_t U, \zeta) \quad (4.22)$$

and finally, combining (4.18) and (4.22), we get

$$\mathcal{E}(\zeta) \leq CT_h^3 \int_{\zeta}^{\zeta+2T_h} \left( \int_{\Omega} \left( \sigma |E|^2 + |\nabla p|^2 \right) dx + \int_{\omega} |A|^2 dx \right) dt + ch \mathcal{G}(\partial_t U, \zeta) . \quad (4.23)$$

We have proved that there exist  $h_o, c, \gamma > 0$  such that for any  $\zeta \geq 0$  and  $h \in (0, h_o]$ , the solution  $(E, H)$  of (1.1) satisfies

$$\mathcal{E}(\zeta) \leq c \frac{1}{h^\gamma} \int_{\zeta}^{\zeta+c \frac{1}{h^\gamma}} \left( \int_{\Omega} \left( \sigma |E|^2 + |\nabla p|^2 \right) dx + \int_{\omega} |A|^2 dx \right) dt + ch \mathcal{G}(\partial_t U, \zeta) . \quad (4.24)$$



By formula (2.1.6) and since  $\mathcal{G}(\partial_t U, \zeta + mc\frac{1}{h^\gamma}) = \mathcal{G}(\partial_t U, \zeta)$  for any  $m$ , this last inequality becomes

$$\begin{aligned} & N\mathcal{E}(\zeta) - \sum_{m=0, \dots, N-1} \int_{\zeta + mc\frac{1}{h^\gamma}}^{\zeta + mc\frac{1}{h^\gamma} + \int_{\Omega} \sigma |E|^2 dx dt} \int_{\Omega} \sigma |E|^2 dx dt \\ &= \sum_{m=0, \dots, N-1} \mathcal{E}(\zeta + mc\frac{1}{h^\gamma}) \\ &\leq c\frac{1}{h^\gamma} \sum_{m=0, \dots, N-1} \int_{\zeta + mc\frac{1}{h^\gamma}}^{\zeta + (m+1)c\frac{1}{h^\gamma}} \left( \int_{\Omega} (\sigma |E|^2 + |\nabla p|^2) dx + \int_{\omega} |A|^2 dx \right) dt + Nch\mathcal{G}(\partial_t U, \zeta) , \end{aligned} \quad (4.25)$$

for any  $N > 1$ . We choose  $N \in (c\frac{1}{h^\gamma}, 1 + c\frac{1}{h^\gamma}]$ . Therefore, there exist  $c, \gamma > 0$  such that for any  $h \in (0, h_o]$ ,

$$\mathcal{E}(\zeta) \leq c \int_{\zeta}^{\zeta + c(\frac{1}{h})^\gamma} \left( \int_{\Omega} (\sigma |E|^2 + |\nabla p|^2) dx + \int_{\omega} |A|^2 dx \right) dt + ch\mathcal{G}(\partial_t U, \zeta) . \quad (4.26)$$

On the other hand, since  $(U, \partial_t U)(\cdot, \zeta) = (A, \partial_t A)(\cdot, \zeta)$ ,

$$\begin{aligned} \mathcal{G}(\partial_t U, \zeta) &= \|\operatorname{curl} E(\cdot, \zeta)\|_{L^2(\Omega)^3}^2 + \|\mu_o \operatorname{curl} H(\cdot, \zeta)\|_{L^2(\Omega)^3}^2 \\ &= \mu_o^2 \|\partial_t H(\cdot, \zeta)\|_{L^2(\Omega)^3}^2 + \|(\partial_t E + \mu_o \sigma E)(\cdot, \zeta)\|_{L^2(\Omega)^3}^2 \\ &\leq c(\mathcal{E}_1(0) + \mathcal{E}(0)) \leq c\|(E_o, H_o)\|_{D(\mathcal{M})}^2 \end{aligned} \quad (4.27)$$

where  $\mathcal{M}$  is the m-accretive operator in  $\mathcal{V}$  with domain  $D(\mathcal{M}) = \mathcal{W}$ , defined as follows.

$$\begin{aligned} \|(F, G)\|_{\mathcal{V}}^2 &= \varepsilon_o \|F\|_{L^2(\Omega)^3}^2 + \mu_o \|G\|_{L^2(\Omega)^3}^2 , \\ \mathcal{M} &= \begin{pmatrix} \frac{1}{\varepsilon_o} \sigma & -\frac{1}{\varepsilon_o} \operatorname{curl} \\ \frac{1}{\mu_o} \operatorname{curl} & 0 \end{pmatrix} . \end{aligned} \quad (4.28)$$

Therefore, combining (4.26) and (4.27), we get the existence of constants  $c, \gamma > 0$  such that for any  $\zeta \geq 0$  and  $h \in (0, h_o]$ ,

$$\mathcal{E}(\zeta) \leq c \int_{\zeta}^{\zeta + c(\frac{1}{h})^\gamma} \left( \int_{\Omega} (\sigma |E|^2 + |\nabla p|^2) dx + \int_{\omega} |A|^2 dx \right) dt + ch\|(E_o, H_o)\|_{D(\mathcal{M})}^2 . \quad (4.29)$$

Denote  $(\mathcal{T}(t))_{t \geq 0}$  the unique semigroup of contractions generated by  $-\mathcal{M}$ . First, suppose that  $(E_o, H_o) \in D(\mathcal{M}^3)$  and let us define the functional of energy

$$\mathcal{E}_2(t) = \frac{1}{2} \int_{\Omega} \left( \varepsilon_o |\partial_t^2 E(x, t)|^2 + \mu_o |\partial_t^2 H(x, t)|^2 \right) dx \quad (4.30)$$

which satisfies

$$\mathcal{E}_2(t_2) - \mathcal{E}_2(t_1) + \int_{t_1}^{t_2} \int_{\Omega} \sigma(x) |\partial_t^2 E(x, t)|^2 dx dt = 0 . \quad (4.31)$$

Let  $X_o = -\mathcal{M}^2(E_o, H_o)$ , then  $(\mathcal{T}(t))_{t \geq 0} X_o = (\partial_t^2 E, \partial_t^2 H)$ ,  $\|\mathcal{T}(\zeta) X_o\|_{\mathcal{V}}^2 = 2\mathcal{E}_2(\zeta)$  and  $\|X_o\|_{D(\mathcal{M})}^2 \leq c\|(E_o, H_o)\|_{D(\mathcal{M}^3)}^2$ . Further, by uniqueness of the orthogonal decomposition in (2.1.1) of Proposition 2.1, (4.29) implies that for any  $(E_o, H_o) \in D(\mathcal{M}^3)$

$$\mathcal{E}_2(\zeta) \leq c \int_{\zeta}^{\zeta + c(\frac{1}{h})^\gamma} \left( \int_{\Omega} (\sigma |\partial_t^2 E|^2 + |\partial_t^2 \nabla p|^2) dx + \int_{\omega} |\partial_t^2 A|^2 dx \right) dt + ch\|(E_o, H_o)\|_{D(\mathcal{M}^3)}^2 . \quad (4.32)$$

Since by Proposition 2.2,  $\mathcal{E}(\zeta) \leq c\mathcal{E}_1(\zeta)$  and in a similar way  $\mathcal{E}_1(\zeta) \leq c\mathcal{E}_2(\zeta)$  for some  $c > 0$ , taking account of the first line of (2.2.1) and (2.2.4), (4.32) becomes

$$\mathcal{E}(\zeta) + \mathcal{E}_1(\zeta) + \mathcal{E}_2(\zeta) \leq c \int_{\zeta}^{\zeta + c(\frac{1}{h})^\gamma} \int_{\Omega} \left( \sigma |E|^2 + \sigma |\partial_t E|^2 + \sigma |\partial_t^2 E|^2 \right) dx dt + ch\|(E_o, H_o)\|_{D(\mathcal{M}^3)}^2 . \quad (4.33)$$

Denote

$$\mathcal{H}(\zeta) = \frac{\mathcal{E}_2(\zeta) + \mathcal{E}_1(\zeta) + \mathcal{E}(\zeta)}{\|(E_o, H_o)\|_{D(\mathcal{M}^3)}^2}. \quad (4.34)$$

Since  $\mathcal{H} \leq 1$ , the inequality (4.33) holds for any  $h > 0$ . Taking  $h = c_0 \mathcal{H}(\zeta)$  with some suitable small constant  $c_0$ , we get the existence of constants  $c, \gamma > 0$  such that for any  $\zeta \geq 0$ ,

$$\mathcal{E}(\zeta) + \mathcal{E}_1(\zeta) + \mathcal{E}_2(\zeta) \leq c \int_{\zeta}^{\zeta + c(\frac{1}{\mathcal{H}(\zeta)})^\gamma} \int_{\Omega} \left( \sigma |E|^2 + \sigma |\partial_t E|^2 + \sigma |\partial_t^2 E|^2 \right) dx dt. \quad (4.35)$$

The function  $\mathcal{H}$  is a continuous positive decreasing real function on  $[0, +\infty)$ , bounded by one and satisfying from (2.1.6), (2.1.7), (4.31) and (4.35),

$$\mathcal{H}(\zeta) \leq c \left( \mathcal{H}(\zeta) - \mathcal{H} \left( \left( \frac{c}{\mathcal{H}(\zeta)} \right)^\gamma + \zeta \right) \right) \quad \forall \zeta \geq 0. \quad (4.36)$$

From [12, p.122, Lemma B], we deduce that there exist  $C, \gamma > 0$  such that for any  $t > 0$

$$\mathcal{E}(t) + \mathcal{E}_1(t) + \mathcal{E}_2(\zeta) \leq \frac{C}{t^\gamma} \|(E_o, H_o)\|_{D(\mathcal{M}^3)}^2 \quad (4.37)$$

that is

$$\|\mathcal{T}(t) Y_o\|_{D(\mathcal{M}^2)}^2 \leq \frac{C}{t^\gamma} \|Y_o\|_{D(\mathcal{M}^3)}^2 \quad \forall Y_o \in D(\mathcal{M}^3). \quad (4.38)$$

Since  $\mathcal{M}$  is an m-accretive operator in  $\mathcal{V}$  with dense domain, one can restrict it to  $D(\mathcal{M}^2)$  in a way that its restriction operator is m-accretive. Thus the following two properties holds.

$$\forall Z_o \in D(\mathcal{M}^2) \quad \exists! Y_o \in D(\mathcal{M}^3) \quad Y_o + \mathcal{M}Y_o = Z_o, \quad (4.39)$$

$$\|Y_o\|_{D(\mathcal{M}^2)} \leq \|Y_o + \mathcal{M}Y_o\|_{D(\mathcal{M}^2)} \quad \forall Y_o \in D(\mathcal{M}^3). \quad (4.40)$$

Consequently,

$$\begin{aligned} \|\mathcal{T}(t) Z_o\|_{D(\mathcal{M})}^2 &= \|\mathcal{T}(t) (Y_o + \mathcal{M}Y_o)\|_{D(\mathcal{M})}^2 \quad \text{by (4.39)} \\ &\leq C_1 \|\mathcal{T}(t) Y_o\|_{D(\mathcal{M}^2)}^2 \\ &\leq \frac{C_2}{t^\gamma} \|Y_o\|_{D(\mathcal{M}^3)}^2 \quad \text{by (4.38)} \\ &\leq \frac{C_3}{t^\gamma} \left( \|Y_o\|_{D(\mathcal{M}^2)}^2 + \|Y_o + \mathcal{M}Y_o\|_{D(\mathcal{M}^2)}^2 \right) \\ &\leq \frac{C_4}{t^\gamma} \|Y_o + \mathcal{M}Y_o\|_{D(\mathcal{M}^2)}^2 \quad \text{by (4.40)} \\ &\leq \frac{C_5}{t^\gamma} \|Z_o\|_{D(\mathcal{M}^2)}^2 \quad \text{by (4.39)} \end{aligned} \quad (4.41)$$

for suitable positive constants  $C_1, C_2, C_3, C_4, C_5 > 0$ .

Now, suppose that  $(E_o, H_o) \in D(\mathcal{M})$ . Since  $\mathcal{M}$  is an m-accretive operator in  $\mathcal{V}$  with dense domain, one can restrict it to  $D(\mathcal{M})$  in a way that its restriction operator is m-accretive. Thus the following two properties holds.

$$\forall (E_o, H_o) \in D(\mathcal{M}) \quad \exists! Z_o \in D(\mathcal{M}^2) \quad Z_o + \mathcal{M}Z_o = (E_o, H_o), \quad (4.42)$$

$$\|Z_o\|_{D(\mathcal{M})} \leq \|Z_o + \mathcal{M}Z_o\|_{D(\mathcal{M})} \quad \forall Z_o \in D(\mathcal{M}^2). \quad (4.43)$$

We conclude that

$$\begin{aligned} \mathcal{E}(t) = \|\mathcal{T}(t) (E_o, H_o)\|_{\mathcal{V}}^2 &= \|\mathcal{T}(t) (Z_o + \mathcal{M}Z_o)\|_{\mathcal{V}}^2 \quad \text{by (4.42)} \\ &\leq C_1 \|\mathcal{T}(t) Z_o\|_{D(\mathcal{M})}^2 \\ &\leq \frac{C_2}{t^\gamma} \|Z_o\|_{D(\mathcal{M}^2)}^2 \quad \text{by (4.41)} \\ &\leq \frac{C_3}{t^\gamma} \left( \|Z_o\|_{D(\mathcal{M})}^2 + \|Z_o + \mathcal{M}Z_o\|_{D(\mathcal{M})}^2 \right) \\ &\leq \frac{C_4}{t^\gamma} \|Z_o + \mathcal{M}Z_o\|_{D(\mathcal{M})}^2 \quad \text{by (4.43)} \\ &\leq \frac{C_5}{t^\gamma} \|(E_o, H_o)\|_{D(\mathcal{M})}^2 \quad \text{by (4.42)} \\ &\leq \frac{C_6}{t^\gamma} (\mathcal{E}(0) + \mathcal{E}_1(0)) \end{aligned} \quad (4.44)$$

for suitable positive constants  $C_1, C_2, C_3, C_4, C_5, C_6 > 0$ .

## 5 Proof of Proposition 4.1

Recall that the definition of  $\omega_o$  and the solution  $U$  were given in Section 4.

Notice that the hypothesis saying that  $\Upsilon \subset (\mathbb{R}^2 \setminus D(m_1, m_2)) \times \mathbb{R}$  implies that  $C_0^\infty(B(x_o, r_o/2)) \subset C_0^\infty(\Omega)$  for any  $x_o \in \overline{\omega_o}$ , where  $B(x_o, r)$  denotes the ball of center  $x_o$  and radius  $r$ .

Let  $\ell \in C^\infty(\mathbb{R}^3)$  be such that  $0 \leq \ell(x) \leq 1$ ,  $\ell = 1$  in  $\mathbb{R}^3 \setminus \Theta$ ,  $\ell(x) \geq \ell_o > 0$  for any  $x \in \overline{\omega_o}$ ,  $\ell = \partial_\nu \ell = 0$  on  $\Upsilon$  and both  $\nabla \ell$  and  $\Delta \ell$  have support in  $\Theta$ .

The proof of Proposition 4.1 comes from the following result.

**Proposition 5.1 -.** *There exists  $c > 0$  such that for any  $x_o \in \overline{\omega_o}$  and any  $h \in (0, 1]$ ,  $L \geq 1$ ,  $\lambda \geq 1$ , the solution  $U$  of (4.1) satisfies*

$$\begin{aligned} & \int_{\Omega \times \mathbb{R}} \chi(x) e^{-\frac{1}{2}(\frac{1}{h}|x-x_o|^2+t^2)} \ell(x) |\partial_t U(x, t)|^2 dx dt \\ & \leq c \left[ \sqrt{\frac{1}{\lambda}} + \left[ \frac{1}{\sqrt{L}} + (1 + hL\lambda) e^{-\frac{1}{ch}} \right] \sqrt{\lambda} \left( \frac{\lambda^2}{\sqrt{h}} + \frac{1}{h} \right) \right] \mathcal{G}(\partial_t U, 0) \\ & \quad + c \left[ h \left( 1 + \sqrt{hL} \right) \sqrt{\lambda} \left( \frac{\lambda^2}{\sqrt{h}} + \frac{1}{h} \right) \right]^2 \sqrt{\lambda} \|(U, \partial_t U)\|_{L^2(\omega \times (-1-T, T+1))}^2 \end{aligned} \quad (5.1)$$

where  $\chi \in C_0^\infty(B(x_o, r_o/2))$ ,  $0 \leq \chi \leq 1$  and

$$T = 4 \left[ \frac{\lambda h L}{\sqrt{2}} + \sqrt{h} L + \frac{\sqrt{2}}{\sqrt{h}} \right]. \quad (5.2)$$

We shall leave the proof of Proposition 5.1 till later (see Section 6). Now we turn to prove Proposition 4.1.

Let  $h \in (0, h_o]$  where  $h_o = \min(1, (r_o/8)^2)$ . We begin by covering  $\overline{\omega_o}$  with a finite collection of balls  $B(x_o^i, 2\sqrt{h})$  for  $i \in I$  with  $x_o^i \in \overline{\omega_o}$  and where  $I$  is a countable set such that the number of elements of  $I$  is  $\frac{c_o}{h\sqrt{h}}$  for some constant  $c_o > 0$  independent of  $h$ . Then, for each  $x_o^i$ , we introduce  $\chi_{x_o^i} \in C_0^\infty(B(x_o^i, r_o/2)) \subset C_0^\infty(\Omega)$  be such that  $0 \leq \chi_{x_o^i} \leq 1$  and  $\chi_{x_o^i} = 1$  on  $B(x_o^i, r_o/4) \supset B(x_o^i, 2\sqrt{h})$ . Consequently, for any  $T_o > 0$ ,

$$\begin{aligned} \int_0^{T_o} \int_{\omega_o} |\partial_t U|^2 dx dt & \leq \frac{1}{\ell_o} e^{\frac{1}{2}T_o^2} \int_0^{T_o} \int_{\omega_o} e^{-\frac{1}{2}t^2} \ell(x) |\partial_t U(x, t)|^2 dx dt \\ & \leq \frac{1}{\ell_o} e^{\frac{1}{2}T_o^2+2} \sum_{i \in I} \int_0^{T_o} \int_{B(x_o^i, 2\sqrt{h})} \chi_{x_o^i}(x) e^{-\frac{1}{2}(\frac{1}{h}|x-x_o^i|^2+t^2)} \ell(x) |\partial_t U(x, t)|^2 dx dt \\ & \leq \frac{1}{\ell_o} e^{\frac{1}{2}T_o^2+2} \sum_{i \in I} \int_{\Omega \times \mathbb{R}} \chi_{x_o^i}(x) e^{-\frac{1}{2}(\frac{1}{h}|x-x_o^i|^2+t^2)} \ell(x) |\partial_t U(x, t)|^2 dx dt. \end{aligned} \quad (5.3)$$

By virtue of Proposition 5.1,  $\int_{\Omega \times \mathbb{R}} \chi_{x_o^i}(x) e^{-\frac{1}{2}(\frac{1}{h}|x-x_o^i|^2+t^2)} \ell(x) |\partial_t U(x, t)|^2 dx dt$  is bounded independently of  $x_o^i$  and it implies that for some constant  $c > 0$ ,

$$\begin{aligned} \int_0^{T_o} \int_{\omega_o} |\partial_t U|^2 dx dt & \leq c \frac{1}{h\sqrt{h}} \left[ \sqrt{\frac{1}{\lambda}} + \left[ \frac{1}{\sqrt{L}} + (1 + hL\lambda) e^{-\frac{1}{ch}} \right] \sqrt{\lambda} \left( \frac{\lambda^2}{\sqrt{h}} + \frac{1}{h} \right) \right] \mathcal{G}(\partial_t U, 0) \\ & \quad + c \frac{1}{h\sqrt{h}} \left[ h \left( 1 + \sqrt{hL} \right) \sqrt{\lambda} \left( \frac{\lambda^2}{\sqrt{h}} + \frac{1}{h} \right) \right]^2 \sqrt{\lambda} \|(U, \partial_t U)\|_{L^2(\omega \times (-1-T, T+1))}^2. \end{aligned} \quad (5.4)$$

First, we choose  $\lambda \geq 1$  be such that  $\lambda = \left(\frac{h_o}{h}\right)^5$  in order that  $\frac{1}{h\sqrt{h}}\frac{1}{\sqrt{\lambda}} \leq Ch$ , then there exist  $c, \delta > 0$  such that for any  $h \in (0, h_o]$ ,

$$\begin{aligned} \int_0^{T_o} \int_{\omega_o} |\partial_t U|^2 dxdt &\leq ch\mathcal{G}(\partial_t U, 0) + c\frac{1}{h^\delta} \left[ \frac{1}{\sqrt{L}} + Le^{-\frac{1}{ch}} \right] \mathcal{G}(\partial_t U, 0) \\ &\quad + c\frac{L}{h^\delta} \|(U, \partial_t U)\|_{L^2(\omega \times (-1-T, T+1))}^2 . \end{aligned} \quad (5.5)$$

Next, we choose  $L \geq 1$  be such that  $L = \left(\frac{h_o}{h}\right)^{2(\delta+1)}$  in order that  $\frac{1}{h^\delta}\frac{1}{\sqrt{L}} \leq Ch$ , then there exist  $c, c', \gamma > 0$  such that for any  $h \in (0, h_o]$ ,

$$T = 4 \left[ \frac{\lambda h L}{\sqrt{2}} + \sqrt{h} L + \frac{\sqrt{2}}{\sqrt{h}} \right] \leq c' \frac{1}{h^\gamma} , \quad (5.6)$$

and further,

$$\int_0^{T_o} \int_{\omega_o} |\partial_t U|^2 dxdt \leq c\frac{1}{h^\gamma} \int_{-\frac{1}{h^\gamma}}^{\frac{1}{h^\gamma}} \int_{\omega} \left( |\partial_t U|^2 + |U|^2 \right) dxdt + ch\mathcal{G}(\partial_t U, 0) . \quad (5.7)$$

By a translation in time, Proposition 4.1 follows.

## 6 Proof of Proposition 5.1

Let  $h \in (0, 1]$ ,  $x_o \in \overline{\omega_o}$ ,  $\chi \in C_0^\infty(B(x_o, r_o/2))$  be such that  $0 \leq \chi \leq 1$  and  $U$  be a solution of (4.1).

By integrations by parts on the time variable, we can check that

$$\begin{aligned} &\int_{\Omega \times \mathbb{R}} \chi(x) e^{-\frac{1}{2}(\frac{1}{h}|x-x_o|^2+t^2)} \ell(x) |\partial_t U(x, t)|^2 dxdt \\ &\leq 2 \left| \int_{\Omega \times \mathbb{R}} \chi(x) e^{-\frac{1}{2}(\frac{1}{h}|x-x_o|^2+\frac{1}{2}t^2)} \ell(x) |U(x, t)|^2 dxdt \right| \\ &\quad + \left| \int_{\Omega \times \mathbb{R}} \chi(x) e^{-\frac{1}{2}(\frac{1}{h}|x-x_o|^2+t^2)} \partial_t^2 U(x, t) \cdot \ell(x) U(x, t) dxdt \right| . \end{aligned} \quad (6.1)$$

Let us introduce for any  $\theta \in \{1, 2\}$ ,

$$a_{o,\theta}(x, t) = e^{-\frac{1}{4}(\frac{1}{h}|x-x_o|^2+\frac{1}{\theta}t^2)} \text{ and } \varphi_\theta(x, t) = \chi(x) a_{o,\theta}(x, t) . \quad (6.2)$$

By the Fourier inversion formula,

$$\begin{aligned}
& 2 \left| \int_{\Omega \times \mathbb{R}} \chi(x) e^{-\frac{1}{2}(\frac{1}{h}|x-x_o|^2 + \frac{1}{2}t^2)} \ell(x) |U(x, t)|^2 dx dt \right| \\
& + \left| \int_{\Omega \times \mathbb{R}} \chi(x) e^{-\frac{1}{2}(\frac{1}{h}|x-x_o|^2 + t^2)} \partial_t^2 U(x, t) \cdot \ell(x) U(x, t) dx dt \right| \\
& = 2 \left| \int_{\Omega \times \mathbb{R}} \varphi_2(x, t) a_{o,2}(x, t) \ell(x) |U(x, t)|^2 dx dt \right| \\
& + \left| \int_{\Omega \times \mathbb{R}} \varphi_1(x, t) a_{o,1}(x, t) \partial_t^2 U(x, t) \cdot \ell(x) U(x, t) dx dt \right| \\
& = 2 \left| \int_{\Omega \times \mathbb{R}} \left( \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} e^{i(x\xi+t\tau)} \widehat{\varphi_2 U}(\xi, \tau) d\xi d\tau \right) \cdot a_{o,2}(x, t) \ell(x) U(x, t) dx dt \right| \\
& + \left| \int_{\Omega \times \mathbb{R}} \left( \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} e^{i(x\xi+t\tau)} \widehat{\varphi_1 \partial_t^2 U}(\xi, \tau) d\xi d\tau \right) \cdot a_{o,1}(x, t) \ell(x) U(x, t) dx dt \right| \\
& \leq 2 \left| \int_{\Omega \times \mathbb{R}} \left( \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{|\tau| < \lambda} e^{i(x\xi+t\tau)} \widehat{\varphi_2 U}(\xi, \tau) d\xi d\tau \right) \cdot a_{o,2}(x, t) \ell(x) U(x, t) dx dt \right| \\
& + \left| \int_{\Omega \times \mathbb{R}} \left( \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{|\tau| < \lambda} e^{i(x\xi+t\tau)} \widehat{\varphi_1 \partial_t^2 U}(\xi, \tau) d\xi d\tau \right) \cdot a_{o,1}(x, t) \ell(x) U(x, t) dx dt \right| \\
& + 2 \left| \int_{\Omega \times \mathbb{R}} \left( \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{|\tau| \geq \lambda} e^{i(x\xi+t\tau)} \widehat{\varphi_2 U}(\xi, \tau) d\xi d\tau \right) \cdot a_{o,2}(x, t) \ell(x) U(x, t) dx dt \right| \\
& + \left| \int_{\Omega \times \mathbb{R}} \left( \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{|\tau| \geq \lambda} e^{i(x\xi+t\tau)} \widehat{\varphi_1 \partial_t^2 U}(\xi, \tau) d\xi d\tau \right) \cdot a_{o,1}(x, t) \ell(x) U(x, t) dx dt \right|
\end{aligned} \tag{6.3}$$

for any  $\lambda \geq 1$ . Here we recall that

$$\widehat{F}(\xi, \tau) = \int_{\mathbb{R}^4} e^{-i(x\xi+t\tau)} F(x, t) dx dt \quad \text{and} \quad F(x, t) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} e^{i(x\xi+t\tau)} \widehat{F}(\xi, \tau) d\xi d\tau \tag{6.4}$$

when  $F$  and  $\widehat{F}$  belong to  $L^1(\mathbb{R}^4)^3$ . On the other hand, from (A1) of Appendix A, we have that

$$\begin{aligned}
& 2 \left| \int_{\Omega \times \mathbb{R}} \left( \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{|\tau| \geq \lambda} e^{i(x\xi+t\tau)} \widehat{\varphi_2 U}(\xi, \tau) d\xi d\tau \right) \cdot a_{o,2}(x, t) \ell(x) U(x, t) dx dt \right| \\
& + \left| \int_{\Omega \times \mathbb{R}} \left( \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{|\tau| \geq \lambda} e^{i(x\xi+t\tau)} \widehat{\varphi_1 \partial_t^2 U}(\xi, \tau) d\xi d\tau \right) \cdot a_{o,1}(x, t) \ell(x) U(x, t) dx dt \right| \\
& \leq c \sqrt{\frac{1}{\lambda}} \mathcal{G}(\partial_t U, 0) .
\end{aligned} \tag{6.5}$$

It remains to study the following two quantities

$$\int_{\Omega \times \mathbb{R}} \left( \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{|\tau| < \lambda} e^{i(x\xi+t\tau)} \widehat{\varphi_2 U}(\xi, \tau) d\xi d\tau \right) \cdot a_{o,2}(x, t) \ell(x) U(x, t) dx dt \tag{6.6}$$

and

$$\int_{\Omega \times \mathbb{R}} \left( \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{|\tau| < \lambda} e^{i(x\xi+t\tau)} \widehat{\varphi_1 \partial_t^2 U}(\xi, \tau) d\xi d\tau \right) \cdot a_{o,1}(x, t) \ell(x) U(x, t) dx dt . \tag{6.7}$$

We claim that

$$\begin{aligned}
& 2 \left| \int_{\Omega \times \mathbb{R}} \left( \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{|\tau| < \lambda} e^{i(x\xi+t\tau)} \widehat{\varphi_2 U}(\xi, \tau) d\xi d\tau \right) \cdot a_{o,2}(x, t) \ell(x) U(x, t) dx dt \right| \\
& + \left| \int_{\Omega \times \mathbb{R}} \left( \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{|\tau| < \lambda} e^{i(x\xi+t\tau)} \widehat{\varphi_1 \partial_t^2 U}(\xi, \tau) d\xi d\tau \right) \cdot a_{o,1}(x, t) \ell(x) U(x, t) dx dt \right| \\
& \leq c \left[ \left( 1 + hL\lambda \right) e^{-\frac{1}{ch}} + \frac{1}{\sqrt{L}} \right] \sqrt{\lambda} \left( \frac{\lambda^2}{\sqrt{h}} + \frac{1}{h} \right) \mathcal{G}(\partial_t U, 0) \\
& + ch \left( 1 + \sqrt{hL} \right) \left( \| (U, \partial_t U) \|_{L^2(\omega \times (-1-T, T+1))^6} \right) \sqrt{\lambda} \left( \frac{\lambda^2}{\sqrt{h}} + \frac{1}{h} \right) \sqrt{\mathcal{G}(\partial_t U, 0)}
\end{aligned} \tag{6.8}$$

with  $T$  given by (5.2) which implies Proposition 5.1 using (6.1), (6.5) and Cauchy-Schwarz inequality.

The proof of our claim is divided into nine subsections. In the next subsection, we introduce suitable sequences of Fourier integral operators. First, we add a new variable  $s \in [0, L]$ . Next, we construct a particular solution of the equation (6.1.10) below for  $(x, t, s) \in \mathbb{R}^4 \times [0, L]$  with good properties on  $\Gamma_1 \cup \Gamma_2$ .

## 6.1 Fourier integral operators

Let  $\varphi \in C_0^\infty(\Omega)$  and  $f = f(x, t) \in L^\infty(\mathbb{R}; L^2(\mathbb{R}^3))$  be such that  $\widehat{\varphi f} \in L^1(\mathbb{R}^4)$ . Let  $h \in (0, 1]$ ,  $L \geq 1$ ,  $\lambda \geq 1$  and  $(x_o, \xi_{o3}) \in \overline{\omega_o} \times (2\mathbb{Z} + 1)$ . Denote  $x = (x_1, x_2, x_3)$  and  $x_o = (x_{o1}, x_{o2}, x_{o3})$ . First, let us introduce for any  $(x, t, s) \in \mathbb{R}^4 \times [0, L]$  and  $n \in \mathbb{Z}$ ,

$$\begin{aligned} & (\mathcal{A}(x_o, \xi_{o3}, n)f)(x, t, s) \\ &= \frac{(-1)^n}{(2\pi)^4} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} e^{i(x_1 \xi_1 + x_2 \xi_2 + t\tau)} e^{i[(-1)^n x_3 + 2n \frac{\xi_{o3}}{|\xi_{o3}|} \rho] \xi_3} e^{-i(|\xi|^2 - \tau^2)hs} \widehat{\varphi f}(\xi, \tau) \\ & \quad a_\theta \left( x_1 - x_{o1} - 2\xi_1 hs, x_2 - x_{o2} - 2\xi_2 hs, (-1)^n x_3 + 2n \frac{\xi_{o3}}{|\xi_{o3}|} \rho - x_{o3} - 2\xi_3 hs, t + 2\tau hs, s \right) d\xi d\tau \end{aligned} \quad (6.1.1)$$

where  $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^2 \times [\xi_{o3} - 1, \xi_{o3} + 1]$ ,

$$a_\theta(x, t, s) = \left( \frac{1}{(is + 1)^{3/2}} e^{-\frac{1}{4h} \frac{|x|^2}{is+1}} \right) \left( \frac{\sqrt{\theta}}{\sqrt{-ihs + \theta}} e^{-\frac{1}{4} - \frac{t^2}{-ihs + \theta}} \right) \quad \forall \theta \in \{1, 2\}. \quad (6.1.2)$$

Next, let us introduce for any  $(x, t, s) \in \mathbb{R}^4 \times [0, L]$ ,

$$(\mathbb{A}(x_o, \xi_{o3})f)(x, t, s) = \sum_{n=-2Q}^{2P+1} (\mathcal{A}(x_o, \xi_{o3}, n)f)(x, t, s), \quad (6.1.3)$$

$$(\mathbb{B}(x_o, \xi_{o3})f)(x, t, s) = \sum_{n=-2Q}^{2P+1} (-1)^n (\mathcal{A}(x_o, \xi_{o3}, n)f)(x, t, s), \quad (6.1.4)$$

where  $(P, Q) \in \mathbb{N}^2$  is the first couple of integer numbers satisfying

$$\begin{cases} P \geq \frac{1}{4\rho} \left( \sqrt{(|\xi_{o3}| + 2)(L^2 + 1)} + 2(|\xi_{o3}| + 1)L \right), \\ Q \geq \frac{1}{4\rho} \left( \sqrt{(|\xi_{o3}| + 2)(L^2 + 1)} + 2(\rho - r_o) \right). \end{cases} \quad (6.1.5)$$

We check after a lengthy but straightforward calculation that for any  $(x, t, s) \in \mathbb{R}^4 \times [0, L]$ ,

$$\begin{cases} (i\partial_s + h(\Delta - \partial_t^2))(\mathbb{A}(x_o, \xi_{o3})f)(x, t, s) = 0, \\ (i\partial_s + h(\Delta - \partial_t^2))(\mathbb{B}(x_o, \xi_{o3})f)(x, t, s) = 0, \end{cases} \quad (6.1.6)$$

and that for any  $(x_1, x_2, t, s) \in \mathbb{R}^3 \times [0, L]$ ,

$$\begin{cases} (\mathbb{A}(x_o, \xi_{o3})f)\left(x_1, x_2, \frac{\xi_{o3}}{|\xi_{o3}|} \rho, t, s\right) = 0, \\ (\mathbb{A}(x_o, \xi_{o3})f)\left(x_1, x_2, -\frac{\xi_{o3}}{|\xi_{o3}|} \rho, t, s\right) = (\mathcal{A}(x_o, \xi_{o3}, -2Q)f)\left(x_1, x_2, -\frac{\xi_{o3}}{|\xi_{o3}|} \rho, t, s\right) \\ \quad + (\mathcal{A}(x_o, \xi_{o3}, 2P+1)f)\left(x_1, x_2, -\frac{\xi_{o3}}{|\xi_{o3}|} \rho, t, s\right), \end{cases} \quad (6.1.7)$$

$$\begin{cases} \partial_{x_3}(\mathbb{B}(x_o, \xi_{o3})f)\left(x_1, x_2, \frac{\xi_{o3}}{|\xi_{o3}|} \rho, t, s\right) = 0, \\ \partial_{x_3}(\mathbb{B}(x_o, \xi_{o3})f)\left(x_1, x_2, -\frac{\xi_{o3}}{|\xi_{o3}|} \rho, t, s\right) = \partial_{x_3}(\mathcal{A}(x_o, \xi_{o3}, -2Q)f)\left(x_1, x_2, -\frac{\xi_{o3}}{|\xi_{o3}|} \rho, t, s\right) \\ \quad - \partial_{x_3}(\mathcal{A}(x_o, \xi_{o3}, 2P+1)f)\left(x_1, x_2, -\frac{\xi_{o3}}{|\xi_{o3}|} \rho, t, s\right). \end{cases} \quad (6.1.8)$$

Let  $f_j = f_j(x, t) \in L^\infty(\mathbb{R}; L^2(\mathbb{R}^3))$  be such that  $\widehat{\varphi f_j} \in L^1(\mathbb{R}^4)$  for any  $j \in \{1, 2, 3\}$ . Let us introduce

$$F = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \quad \text{and} \quad V(x_o, \xi_{o3}) F = \begin{pmatrix} \mathbb{A}(x_o, \xi_{o3}) f_1 \\ \mathbb{A}(x_o, \xi_{o3}) f_2 \\ \mathbb{B}(x_o, \xi_{o3}) f_3 \end{pmatrix} \quad (6.1.9)$$

then

$$(i\partial_s + h(\Delta - \partial_t^2))(V(x_o, \xi_{o3}) F)(x, t, s) = 0 \quad \forall (x, t, s) \in \mathbb{R}^4 \times [0, L]. \quad (6.1.10)$$

On another hand, let  $U$  be the solution of (4.1). Denote

$$U = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad \text{then} \quad \begin{cases} \forall j \in \{1, 2, 3\} & \partial_t^2 u_j - \Delta u_j = 0 & \text{in } \Omega \times \mathbb{R} \\ & u_1 = u_2 = 0 & \text{on } (\Gamma_1 \cup \Gamma_2) \times \mathbb{R} \\ & \partial_{x_3} u_3 = 0 & \text{on } (\Gamma_1 \cup \Gamma_2) \times \mathbb{R} \end{cases} \quad (6.1.11)$$

because  $\operatorname{div} U = 0$  and  $U \times \nu = 0$ . Further, by (4.3) and (4.6),

$$\exists c > 0 \quad \|u_j(\cdot, t)\|_{H^1(\Omega)}^2 + \|\partial_t u_j(\cdot, t)\|_{L^2(\Omega)}^2 \leq c\mathcal{G}(U, 0) \quad \forall j \in \{1, 2, 3\}. \quad (6.1.12)$$

By multiplying the equation (6.1.10) by  $\ell(x)U(x, t)$  and integrating by parts over  $\Omega \times [-T, T] \times [0, L]$ , we have that for all  $(x_o, \xi_{o3}) \in \overline{\omega_o} \times (2\mathbb{Z} + 1)$  and all  $h \in (0, 1]$ ,  $L \geq 1$ ,  $T > 0$ ,

$$\begin{aligned} 0 = & -i \int_{\Omega} \int_{-T}^T (V(x_o, \xi_{o3}) F)(\cdot, \cdot, 0) \cdot \ell U dx dt \\ & + i \int_{\Omega} \int_{-T}^T (V(x_o, \xi_{o3}) F)(\cdot, \cdot, L) \cdot \ell U dx dt \\ & - h \int_{\Gamma_1 \cup \Gamma_2} \int_{-T}^T \left\{ \left( \int_0^L \mathbb{A}(x_o, \xi_{o3}) f_1 ds \right) \ell \partial_\nu u_1 + \left( \int_0^L \mathbb{A}(x_o, \xi_{o3}) f_2 ds \right) \ell \partial_\nu u_2 \right\} d\sigma dt \\ & + h \int_{\Gamma_1 \cup \Gamma_2} \int_{-T}^T \left( \int_0^L \partial_\nu (\mathbb{B}(x_o, \xi_{o3}) f_3) ds \right) \ell u_3 d\sigma dt \\ & - h \int_{\int_{(\Gamma_1 \cup \Gamma_2) \cap \Theta} \int_{-T}^T \left( \int_0^L \mathbb{B}(x_o, \xi_{o3}) f_3 ds \right) \partial_\nu \ell u_3 d\sigma dt} \\ & - h \int_{\Omega} \left[ \left( \int_0^L \partial_t (V(x_o, \xi_{o3}) F)(\cdot, t, \cdot) ds \right) \cdot \ell U(\cdot, t) - \left( \int_0^L (V(x_o, \xi_{o3}) F)(\cdot, t, \cdot) ds \right) \cdot \ell \partial_t U(\cdot, t) \right]_{-T}^T dx \\ & + h \int_{\omega} \int_{-T}^T \left( \int_0^L V(x_o, \xi_{o3}) F ds \right) \cdot [2(\nabla \ell \cdot \nabla) U + \Delta \ell U] dx dt \\ \equiv & \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5 + \mathcal{I}_6 + \mathcal{I}_7. \end{aligned} \quad (6.1.13)$$

The different terms of the last equality will be estimated separately. The quantity  $\mathcal{I}_1$  will allow us to recover (6.6) (resp. (6.7)) when  $\theta = 2$  and  $\varphi F = \varphi_2 U$  (resp. when  $\theta = 1$  and  $\varphi F = \varphi_1 \partial_t^2 U$ ). The dispersion property for the one dimensional Schrödinger operator will be used for making  $\mathcal{I}_2$  small for large  $L$ . We treat  $\mathcal{I}_3$  (resp.  $\mathcal{I}_4$ ) by applying the formula (6.1.7) (resp. (6.1.8)). The quantity  $\mathcal{I}_5$  and  $\mathcal{I}_7$  will correspond to a term localized in  $\omega$ . Finally, an appropriate choice of  $T$  will bound  $\mathcal{I}_6$  and give the desired inequality (6.9.2) below.

## 6.2 Estimate for $\mathcal{I}_1$ (the term at $s = 0$ )

We estimate  $\mathcal{I}_1 = -i \int_{\Omega} \int_{-T}^T (V(x_o, \xi_{o3}) F)(x, t, 0) \cdot \ell(x) U(x, t) dx dt$  as follows.

**Lemma 6.1 .-** *There exists  $c > 0$  such that for any  $(x_o, \xi_{o3}) \in \overline{\omega_o} \times (2\mathbb{Z} + 1)$  and  $h \in (0, 1]$ ,  $\lambda \geq 1$ ,  $T > 0$ , we have*

$$\begin{aligned} & \left| \mathcal{I}_1 + i \int_{\Omega \times \mathbb{R}} \left( \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} e^{i(x\xi + t\tau)} \widehat{\varphi F}(\xi, \tau) d\xi d\tau \right) \cdot a_{o,\theta}(x, t) \ell(x) U(x, t) dx dt \right| \\ & \leq c \left( e^{-\frac{1}{ch}} + e^{-\frac{1}{8}T^2} \right) \sqrt{\mathcal{G}(U, 0)} \left( \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} \left| \widehat{\varphi F}(\xi, \tau) \right| d\xi d\tau \right). \end{aligned} \quad (6.2.1)$$

Proof .- We start with the third component of  $V(x_o, \xi_{o3})F$ . First,

$$\begin{aligned} & (\mathbb{B}(x_o, \xi_{o3})f)(x, t, 0) \\ & = \sum_{n=-2Q}^{2P+1} \left[ a_\theta \left( x_1 - x_{o1}, x_2 - x_{o2}, (-1)^n x_3 + 2n \frac{\xi_{o3}}{|\xi_{o3}|} \rho - x_{o3}, t, 0 \right) \right. \\ & \quad \left. \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} e^{i(x_1\xi_1 + x_2\xi_2 + t\tau)} e^{i[(-1)^n x_3 + 2n \frac{\xi_{o3}}{|\xi_{o3}|} \rho] \xi_3} \widehat{\varphi f}(\xi, \tau) d\xi d\tau \right] \\ & = a_{o,\theta}(x, t) \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} e^{i(x\xi + t\tau)} \widehat{\varphi f}(\xi, \tau) d\xi d\tau \\ & \quad + \sum_{n \in \{-2Q, \dots, 2P+1\} \setminus \{0\}} \left[ a_\theta \left( x_1 - x_{o1}, x_2 - x_{o2}, (-1)^n x_3 + 2n \frac{\xi_{o3}}{|\xi_{o3}|} \rho - x_{o3}, t, 0 \right) \right. \\ & \quad \left. \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} e^{i(x_1\xi_1 + x_2\xi_2 + t\tau)} e^{i[(-1)^n x_3 + 2n \frac{\xi_{o3}}{|\xi_{o3}|} \rho] \xi_3} \widehat{\varphi f}(\xi, \tau) d\xi d\tau \right]. \end{aligned} \quad (6.2.2)$$

Next, we estimate the discrete sum over  $\{-2Q, \dots, 2P+1\} \setminus \{0\}$ .

$$\begin{aligned} & \left| \sum_{n \in \{-2Q, \dots, 2P+1\} \setminus \{0\}} \left[ a_\theta \left( x_1 - x_{o1}, x_2 - x_{o2}, (-1)^n x_3 + 2n \frac{\xi_{o3}}{|\xi_{o3}|} \rho - x_{o3}, t, 0 \right) \right. \right. \\ & \quad \left. \left. \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} e^{i(x_1\xi_1 + x_2\xi_2 + t\tau)} e^{i[(-1)^n x_3 + 2n \frac{\xi_{o3}}{|\xi_{o3}|} \rho] \xi_3} \widehat{\varphi f}(\xi, \tau) d\xi d\tau \right] \right| \\ & \leq \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} \left| \widehat{\varphi f}(\xi, \tau) \right| d\xi d\tau e^{-\frac{t^2}{8}} e^{-\frac{|(x_1 - x_{o1}, x_2 - x_{o2})|^2}{4h}} \sum_{n \in \mathbb{Z} \setminus \{0\}} e^{-\frac{((-1)^n x_3 + 2n \frac{\xi_{o3}}{|\xi_{o3}|} \rho - x_{o3})^2}{4h}}. \end{aligned} \quad (6.2.3)$$

Remark that for any  $(x_{o1}, x_{o2}, x_{o3}) \in \overline{\omega_o}$  and  $(x_1, x_2, x_3) \in \Omega$ , the following two cases appears. If  $(x_1, x_2) \notin \overline{D(m_1 - r_o/2, m_2 - r_o/2)}$  then we have  $(x_1 - x_{o1})^2 + (x_2 - x_{o2})^2 \geq (r_o/2)^2$ . If  $(x_1, x_2) \in \overline{D(m_1 - r_o/2, m_2 - r_o/2)}$ , then  $x_3 \in [-\rho, \rho]$  and we get  $2r_o \leq \pm((-1)^n x_3 - x_{o3}) \frac{\xi_{o3}}{|\xi_{o3}|} + 2\rho$ . Therefore, for some  $c > 0$ ,

$$\begin{aligned} & e^{-\frac{|(x_1 - x_{o1}, x_2 - x_{o2})|^2}{4h}} \sum_{n \in \mathbb{Z} \setminus \{0\}} e^{-\frac{((-1)^n x_3 + 2n \frac{\xi_{o3}}{|\xi_{o3}|} \rho - x_{o3})^2}{4h}} \\ & \leq e^{-\frac{r_o^2}{8h}} \sum_{n \in \mathbb{Z}} e^{-\frac{((-1)^n x_3 + 2n \frac{\xi_{o3}}{|\xi_{o3}|} \rho - x_{o3})^2}{4h}} + 2 \sum_{n \geq 0} e^{-\frac{n^2 \rho^2}{h}} e^{-\frac{r_o^2}{h}} \\ & \leq ce^{-\frac{1}{ch}}. \end{aligned} \quad (6.2.4)$$



Now, we deduce from (6.2.3) and (6.2.4) that

$$\begin{aligned}
& \left| -i \int_{\Omega} \int_{-T}^T (\mathbb{B}(x_o, \xi_{o3}) f)(x, t, 0) \ell(x) u_3(x, t) dx dt \right. \\
& \quad \left. + i \int_{\Omega} \int_{-T}^T \left( \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} e^{i(x\xi+t\tau)} \widehat{\varphi f}(\xi, \tau) d\xi d\tau \right) a_{o,\theta}(x, t) \ell(x) u_3(x, t) dx dt \right| \\
& \leq \frac{c}{(2\pi)^4} e^{-\frac{1}{ch}} \int_{\Omega} \int_{-T}^T e^{-\frac{t^2}{8}} |\ell(x) u_3(x, t)| dx dt \left( \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} |\widehat{\varphi f}(\xi, \tau)| d\xi d\tau \right) \\
& \leq c e^{-\frac{1}{ch}} \sqrt{\mathcal{G}(U, 0)} \left( \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} |\widehat{\varphi f}(\xi, \tau)| d\xi d\tau \right)
\end{aligned} \tag{6.2.5}$$

where in the last line we have used the fact that the solution  $U$  has the following property, from Cauchy-Schwarz inequality and (6.1.12),

$$\begin{aligned}
\int_{-T}^T e^{-\frac{t^2}{8}} \int_{\Omega} |\ell(x) u_3(x, t)| dx dt & \leq c \sqrt{|\Omega|} \left( \int_{-\infty}^{\infty} e^{-\frac{t^2}{8}} dt \right) \sqrt{\mathcal{G}(U, 0)} \\
& \leq c \sqrt{|\Omega|} (2\sqrt{2\pi}) \sqrt{\mathcal{G}(U, 0)}.
\end{aligned} \tag{6.2.6}$$

Here and hereafter,  $c$  will be used to denote a generic constant, not necessarily the same in any two places. On the other hand, we cut the integral on time into two parts to obtain

$$\begin{aligned}
& \int_{\Omega \times \mathbb{R}} \left( \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} e^{i(x\xi+t\tau)} \widehat{\varphi f}(\xi, \tau) d\xi d\tau \right) a_{o,\theta}(x, t) \ell(x) u_3(x, t) dx dt \\
& = \int_{\Omega} \int_{-T}^T \left( \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} e^{i(x\xi+t\tau)} \widehat{\varphi f}(\xi, \tau) d\xi d\tau \right) a_{o,\theta}(x, t) \ell(x) u_3(x, t) dx dt \\
& \quad + \int_{\Omega \setminus \mathbb{R} \setminus (-T, T)} \left( \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} e^{i(x\xi+t\tau)} \widehat{\varphi f}(\xi, \tau) d\xi d\tau \right) e^{-\frac{1}{4}(\frac{1}{h}|x-x_o|^2 + \frac{1}{\theta}t^2)} \ell(x) u_3(x, t) dx dt
\end{aligned} \tag{6.2.7}$$

and, by using (6.1.12) and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
& \left| i \int_{\Omega \setminus \mathbb{R} \setminus (-T, T)} \left( \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} e^{i(x\xi+t\tau)} \widehat{\varphi f}(\xi, \tau) d\xi d\tau \right) e^{-\frac{1}{4}(\frac{1}{h}|x-x_o|^2 + \frac{1}{\theta}t^2)} \ell(x) u_3(x, t) dx dt \right| \\
& \leq c e^{-\frac{1}{8}T^2} \sqrt{\mathcal{G}(U, 0)} \left( \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} |\widehat{\varphi f}(\xi, \tau)| d\xi d\tau \right).
\end{aligned} \tag{6.2.8}$$

We conclude from (6.2.5), (6.2.7) and (6.2.8) that

$$\begin{aligned}
& \left| -i \int_{\Omega} \int_{-T}^T (\mathbb{B}(x_o, \xi_{o3}) f)(x, t, 0) \ell(x) u_3(x, t) dx dt \right. \\
& \quad \left. + i \int_{\Omega \times \mathbb{R}} \left( \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} e^{i(x\xi+t\tau)} \widehat{\varphi f}(\xi, \tau) d\xi d\tau \right) a_{o,\theta}(x, t) \ell(x) u_3(x, t) dx dt \right| \\
& \leq c \left( e^{-\frac{1}{ch}} + e^{-\frac{1}{8}T^2} \right) \sqrt{\mathcal{G}(U, 0)} \left( \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} |\widehat{\varphi f}(\xi, \tau)| d\xi d\tau \right).
\end{aligned} \tag{6.2.9}$$

Similarly,

$$\begin{aligned}
& \left| -i \int_{\Omega} \int_{-T}^T (\mathbb{A}(x_o, \xi_{o3}) f)(x, t, 0) \ell(x) u_j(x, t) dx dt \right. \\
& \quad \left. + i \int_{\Omega \times \mathbb{R}} \left( \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} e^{i(x\xi + t\tau)} \widehat{\varphi} f(\xi, \tau) d\xi d\tau \right) a_{o,\theta}(x, t) \ell(x) u_j(x, t) dx dt \right| \\
& \leq c \left( e^{-\frac{1}{ch}} + e^{-\frac{1}{8}T^2} \right) \sqrt{\mathcal{G}(U, 0)} \left( \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} \left| \widehat{\varphi} f(\xi, \tau) \right| d\xi d\tau \right).
\end{aligned} \tag{6.2.10}$$

This completes the proof.

### 6.3 Estimate for $\mathcal{I}_2$ (the term at $s = L$ )

We estimate  $\mathcal{I}_2 = i \int_{\Omega} \int_{-T}^T (V(x_o, \xi_{o3}) F)(x, t, L) \cdot \ell(x) U(x, t) dx dt$  as follows.

**Lemma 6.2 .-** *There exists  $c > 0$  such that for any  $(x_o, \xi_{o3}) \in \overline{\omega_o} \times (2\mathbb{Z} + 1)$  and  $h \in (0, 1]$ ,  $L \geq 1$ ,  $\lambda \geq 1$ ,  $T > 0$ , we have*

$$|\mathcal{I}_2| \leq c \left( \frac{1}{\sqrt{L}} + e^{-\frac{1}{4h}} \right) \sqrt{\mathcal{G}(U, 0)} \left( \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} \left| \widehat{\varphi} F(\xi, \tau) \right| d\xi d\tau \right). \tag{6.3.1}$$

Proof .- We start with the third component of  $V(x_o, \xi_{o3}) F$ . First,

$$\begin{aligned}
& |(\mathbb{B}(x_o, \xi_{o3}) f)(x, t, L)| \\
& \leq \sum_{n \in \mathbb{Z} \setminus \{-2Q, \dots, 2P+1\}} |(\mathcal{A}(x_o, \xi_{o3}, n) f)(x, t, L)| + \left| \sum_{n \in \mathbb{Z}} (-1)^n (\mathcal{A}(x_o, \xi_{o3}, n) f)(x, t, L) \right|.
\end{aligned} \tag{6.3.2}$$

Next,

$$\begin{aligned}
& \sum_{n \in \mathbb{Z} \setminus \{-2Q, \dots, 2P+1\}} |(\mathcal{A}(x_o, \xi_{o3}, n) f)(x, t, L)| \\
& \leq \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} \left| \widehat{\varphi} f(\xi, \tau) \right| \left( \frac{\sqrt{\theta}}{(\sqrt{(hL)^2 + \theta^2})^{1/2}} e^{-\frac{\theta}{4} \frac{(t+2\tau hL)^2}{(hL)^2 + \theta^2}} \right) \\
& \quad \left( \frac{1}{\sqrt{L^2+1}} e^{-\frac{1}{4h} \frac{|(x_1-x_{o1}-2\xi_1 hL, x_2-x_{o2}-2\xi_2 hL)|^2}{L^2+1}} \right) \\
& \quad \frac{1}{(\sqrt{L^2+1})^{1/2}} \left( \sum_{n \in \mathbb{Z} \setminus \{-2Q, \dots, 2P+1\}} e^{-\frac{1}{4h} \frac{((-1)^n x_3 + 2n \frac{\xi_{o3}}{|\xi_{o3}|} \rho - x_{o3} - 2\xi_3 hL)^2}{L^2+1}} \right) d\xi d\tau.
\end{aligned} \tag{6.3.3}$$

When  $\xi_3 \in (\xi_{o3} - 1, \xi_{o3} + 1)$  with  $\xi_{o3} \in (2\mathbb{Z} + 1)$ ,

$$\begin{aligned}
\sqrt{L^2+1} & \leq 4P\rho - 2(|\xi_{o3}| + 1)L \quad \text{from our choice of } P \\
& \leq 4P\rho - 2|\xi_3| hL + 2\rho - |x_3| - |x_{o3}| \quad \text{because } |x_3| + |x_{o3}| \leq 2(\rho - r_o) \\
& \leq 4P\rho - 2|\xi_3| hL + 2\rho + \frac{\xi_{o3}}{|\xi_{o3}|} [ -(-1)^n x_3 - x_{o3} ] \quad \forall n \in \mathbb{Z}
\end{aligned} \tag{6.3.4}$$

thus

$$\begin{aligned}
& \sum_{n \geq 2P+2} e^{-\frac{1}{4h} \frac{((-1)^n x_3 + 2n \frac{\xi_{o3}}{|\xi_{o3}|} \rho - x_{o3} - 2\xi_3 hL)^2}{L^2+1}} = \sum_{n \geq 2P+2} e^{-\frac{1}{4h} \frac{(2n\rho - 2|\xi_3| hL + \frac{\xi_{o3}}{|\xi_{o3}|} [(-1)^n x_3 - x_{o3}])^2}{L^2+1}} \\
& = \sum_{n \geq 1} e^{-\frac{1}{4h} \frac{(2n\rho + 4P\rho - 2|\xi_3| hL + 2\rho + \frac{\xi_{o3}}{|\xi_{o3}|} [ -(-1)^n x_3 - x_{o3} ])^2}{L^2+1}} \\
& \leq \sum_{n \geq 1} e^{-\frac{1}{4h} \frac{(2n\rho)^2}{L^2+1}} e^{-\frac{1}{4h} \frac{(4P\rho - 2|\xi_3| hL + 2\rho + \frac{\xi_{o3}}{|\xi_{o3}|} [ -(-1)^n x_3 - x_{o3} ])^2}{L^2+1}} \\
& \leq e^{-\frac{1}{4h}} \sum_{n \geq 1} e^{-\frac{1}{h} \frac{(n\rho)^2}{L^2+1}} \leq e^{-\frac{1}{4h}} \left( \frac{\sqrt{\pi}}{2} \frac{\sqrt{h} \sqrt{L^2+1}}{\rho} \right).
\end{aligned} \tag{6.3.5}$$

Also,

$$\begin{aligned}
\sqrt{L^2+1} &\leq 4Q\rho - 2(\rho - r_o) \quad \text{from our choice of } Q \\
&\leq 4Q\rho + 2|\xi_3| hL - |x_3| - |x_{o3}| \quad \text{because } |x_3| + |x_{o3}| \leq 2(\rho - r_o) \\
&\leq 4Q\rho + 2|\xi_3| hL - \frac{\xi_{o3}}{|\xi_{o3}|} [(-1)^n x_3 - x_{o3}] \quad \forall n \in \mathbb{Z}
\end{aligned} \tag{6.3.6}$$

thus

$$\begin{aligned}
&\sum_{n \leq -2Q-1} e^{-\frac{1}{4h} \frac{\left((-1)^n x_3 + 2n \frac{\xi_{o3}}{|\xi_{o3}|} \rho - x_{o3} - 2\xi_3 hL\right)^2}{L^2+1}} = \sum_{n \geq 2Q+1} e^{-\frac{1}{4h} \frac{\left(2n\rho + 2|\xi_3| hL - \frac{\xi_{o3}}{|\xi_{o3}|} [(-1)^n x_3 - x_{o3}]\right)^2}{L^2+1}} \\
&\leq \sum_{n \geq 1} e^{-\frac{1}{4h} \frac{\left(2n\rho + 4Q\rho + 2|\xi_3| hL - \frac{\xi_{o3}}{|\xi_{o3}|} [(-1)^n x_3 - x_{o3}]\right)^2}{L^2+1}} \\
&\leq \sum_{n \geq 1} e^{-\frac{1}{4h} \frac{(2n\rho)^2}{L^2+1}} e^{-\frac{1}{4h} \frac{\left(4Q\rho + 2|\xi_3| hL - \frac{\xi_{o3}}{|\xi_{o3}|} [(-1)^n x_3 - x_{o3}]\right)^2}{L^2+1}} \\
&\leq e^{-\frac{1}{4h}} \sum_{n \geq 1} e^{-\frac{1}{h} \frac{(n\rho)^2}{L^2+1}} \leq e^{-\frac{1}{4h}} \left( \frac{\sqrt{\pi} \sqrt{h} \sqrt{L^2+1}}{\rho} \right).
\end{aligned} \tag{6.3.7}$$

It implies from (6.3.3), (6.3.5) and (6.3.7) that

$$\begin{aligned}
&\sum_{n \in \mathbb{Z} \setminus \{-2Q, \dots, 2P+1\}} |(\mathcal{A}(x_o, \xi_{o3}, n) f)(x, t, L)| \\
&\leq \frac{1}{(2\pi)^4} \frac{1}{\sqrt{L^2+1}} \frac{1}{(\sqrt{L^2+1})^{1/2}} \left( \frac{\sqrt{\pi} \sqrt{h} \sqrt{L^2+1}}{\rho} \right) e^{-\frac{1}{4h}} \\
&\quad \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} |\widehat{\varphi f}(\xi, \tau)| \left( \frac{\sqrt{\theta}}{(\sqrt{(hL)^2 + \theta^2})^{1/2}} e^{-\frac{\theta}{4} \frac{(t+2\tau hL)^2}{(hL)^2 + \theta^2}} \right) d\xi d\tau.
\end{aligned} \tag{6.3.8}$$

Now,

$$\begin{aligned}
&\left| \sum_{n \in \mathbb{Z}} (-1)^n (\mathcal{A}(x_o, \xi_{o3}, n) f)(x, t, L) \right| \\
&\leq \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} |\widehat{\varphi f}(\xi, \tau)| \left( \frac{\sqrt{\theta}}{(\sqrt{(hL)^2 + \theta^2})^{1/2}} e^{-\frac{\theta}{4} \frac{(t+2\tau hL)^2}{(hL)^2 + \theta^2}} \right) \\
&\quad \left( \frac{1}{\sqrt{L^2+1}} e^{-\frac{1}{4h} \frac{|(x_1 - x_{o1} - 2\xi_1 hL, x_2 - x_{o2} - 2\xi_2 hL)|^2}{L^2+1}} \right) \\
&\quad \left| \sum_{n \in \mathbb{Z}} \left( \frac{1}{\sqrt{iL+1}} e^{i[(-1)^n x_3 + 2n \frac{\xi_{o3}}{|\xi_{o3}|} \rho] \xi_3} e^{-\frac{1}{4h} \frac{\left((-1)^n x_3 + 2n \frac{\xi_{o3}}{|\xi_{o3}|} \rho - x_{o3} - 2\xi_3 hL\right)^2}{iL+1}} \right) \right| d\xi d\tau \\
&\leq \frac{c}{\sqrt{L^2+1}} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} |\widehat{\varphi f}(\xi, \tau)| \left( \frac{\sqrt{\theta}}{(\sqrt{(hL)^2 + \theta^2})^{1/2}} e^{-\frac{\theta}{4} \frac{(t+2\tau hL)^2}{(hL)^2 + \theta^2}} \right) d\xi d\tau
\end{aligned} \tag{6.3.9}$$

because from Appendix B with  $z = \frac{4h}{\rho^2} (iL+1)$ , we know that

$$\left| \frac{1}{\sqrt{iL+1}} \sum_{n \in \mathbb{Z}} e^{i[(-1)^n x_3 + 2n \frac{\xi_{o3}}{|\xi_{o3}|} \rho] \xi_3} e^{-\frac{1}{4h} \frac{\left((-1)^n x_3 + 2n \frac{\xi_{o3}}{|\xi_{o3}|} \rho - x_{o3} - 2\xi_3 hL\right)^2}{iL+1}} \right| \leq \frac{2\sqrt{h}}{\rho} \left( \frac{\sqrt{\pi}}{2} + \frac{\rho}{\sqrt{h}} \right). \tag{6.3.10}$$

Finally, (6.3.2), (6.3.8) and (6.3.10) imply that

$$\begin{aligned}
&|(\mathbb{B}(x_o, \xi_{o3}) f)(x, t, L)| \\
&\leq c \left( \frac{1}{\sqrt{L^2+1}} + \frac{1}{(\sqrt{L^2+1})^{1/2}} e^{-\frac{1}{4h}} \right) \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} |\widehat{\varphi f}(\xi, \tau)| \left( \frac{\sqrt{\theta}}{(\sqrt{(hL)^2 + \theta^2})^{1/2}} e^{-\frac{\theta}{4} \frac{(t+2\tau hL)^2}{(hL)^2 + \theta^2}} \right) d\xi d\tau
\end{aligned} \tag{6.3.11}$$

and we conclude that

$$\begin{aligned}
& \left| i \int_{\Omega} \int_{-T}^T (\mathbb{B}(x_o, \xi_{o3}) f)(x, t, L) \ell(x) u_3(x, t) dx dt \right| \\
& \leq c \left( \frac{1}{\sqrt{L^2+1}} + \frac{1}{(\sqrt{L^2+1})^{1/2}} e^{-\frac{1}{4h}} \right) \\
& \quad \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} \left| \widehat{\varphi f}(\xi, \tau) \right| d\xi \int_{-T}^T \left( \frac{\sqrt{\theta}}{(\sqrt{(hL)^2+\theta^2})^{1/2}} e^{-\frac{\theta}{4} \frac{(t+2\tau hL)^2}{(hL)^2+\theta^2}} \right) \int_{\Omega} |\ell(x) u_3(x, t)| dx dt d\tau \\
& \leq c \left( \frac{1}{\sqrt{L}} + e^{-\frac{1}{4h}} \right) \sqrt{\mathcal{G}(U, 0)} \left( \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} \left| \widehat{\varphi f}(\xi, \tau) \right| d\xi d\tau \right)
\end{aligned} \tag{6.3.12}$$

where in the last line we have used the fact that the solution  $U$  has the following property, from Cauchy-Schwarz inequality and (6.1.12),

$$\begin{aligned}
& \int_{-T}^T e^{-\frac{\theta}{4} \frac{(t+2\tau hL)^2}{(hL)^2+\theta^2}} \int_{\Omega} |\ell(x) u_3(x, t)| dx dt \\
& \leq c \sqrt{|\Omega|} \left( \int_{-\infty}^{\infty} e^{-\frac{\theta}{4} \frac{t^2}{(hL)^2+\theta^2}} dt \right) \sqrt{\mathcal{G}(U, 0)} \\
& \leq c \sqrt{|\Omega|} \left( \frac{2\sqrt{\pi}}{\sqrt{\theta}} \sqrt{(hL)^2 + \theta^2} \right) \sqrt{\mathcal{G}(U, 0)}.
\end{aligned} \tag{6.3.13}$$

Similarly,

$$\begin{aligned}
& \left| i \int_{\Omega} \int_{-T}^T (\mathbb{A}(x_o, \xi_{o3}) f)(x, t, L) \ell(x) u_j(x, t) dx dt \right| \\
& \leq c \left( \frac{1}{\sqrt{L}} + e^{-\frac{1}{4h}} \right) \sqrt{\mathcal{G}(U, 0)} \left( \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} \left| \widehat{\varphi f}(\xi, \tau) \right| d\xi d\tau \right),
\end{aligned} \tag{6.3.14}$$

using the estimate

$$\left| (\mathbb{A}(x_o, \xi_{o3}) f)(x, t, L) \right| \leq \sum_{n \in \mathbb{Z} \setminus \{-2Q, \dots, 2P+1\}} |(\mathcal{A}(x_o, \xi_{o3}, n) f)(x, t, L)| + \left| \sum_{n \in \mathbb{Z}} (\mathcal{A}(x_o, \xi_{o3}, n) f)(x, t, L) \right| \tag{6.3.15}$$

and

$$\left| \frac{1}{\sqrt{iL+1}} \sum_{n \in \mathbb{Z}} (-1)^n e^{i \left[ (-1)^n x_3 + 2n \frac{\xi_{o3}}{|\xi_{o3}|} \rho \right] \xi_3} e^{-\frac{1}{4h} \frac{\left( (-1)^n x_3 + 2n \frac{\xi_{o3}}{|\xi_{o3}|} \rho - x_{o3} - 2\xi_3 hL \right)^2}{iL+1}} \right| \leq \frac{2\sqrt{h}}{\rho} \left( \frac{\sqrt{\pi}}{2} + \frac{\rho}{\sqrt{h}} \right) \tag{6.3.16}$$

deduced from Appendix B with  $z = \frac{4h}{\rho^2} (iL+1)$ . This completes the proof.

## 6.4 Estimate for $\mathcal{I}_3$ (the boundary term with $\mathbb{A}$ )

We estimate  $\mathcal{I}_3 = -h \int_{\Gamma_1 \cup \Gamma_2} \int_{-T}^T \left\{ \left( \int_0^L \mathbb{A}(x_o, \xi_{o3}) f_1 ds \right) \ell \partial_{\nu} u_1 + \left( \int_0^L \mathbb{A}(x_o, \xi_{o3}) f_2 ds \right) \ell \partial_{\nu} u_2 \right\} d\sigma dt$  as follows.

**Lemma 6.3 .-** *There exists  $c > 0$  such that for any  $(x_o, \xi_{o3}) \in \overline{\omega_o} \times (2\mathbb{Z}+1)$  and  $h \in (0, 1]$ ,  $L \geq 1$ ,  $\lambda \geq 1$ ,  $T > 0$ , we have*

$$|\mathcal{I}_3| \leq chL e^{-\frac{1}{4h}} \sqrt{\mathcal{G}(U, 0)} \left( \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} \left| \widehat{\varphi f}(\xi, \tau) \right| d\xi d\tau \right). \tag{6.4.1}$$

Proof .- First, by (6.1.7), we deduce that

$$\begin{aligned}
& \int_{\Gamma_1 \cup \Gamma_2} (\mathbb{A}(x_o, \xi_{o3}) f)(x, t, s) \ell(x) \partial_{x_3} u_j(x, t) d\sigma \\
& \leq \int_{\Gamma_1 \cup \Gamma_2} \left| (\mathcal{A}(x_o, \xi_{o3}, -2Q) f) \left( x_1, x_2, -\frac{\xi_{o3}}{|\xi_{o3}|} \rho, t, s \right) \right| \left| \ell \partial_{x_3} u_j \left( x_1, x_2, -\frac{\xi_{o3}}{|\xi_{o3}|} \rho, t \right) \right| d\sigma \\
& \quad + \int_{\Gamma_1 \cup \Gamma_2} \left| (\mathcal{A}(x_o, \xi_{o3}, 2P+1) f) \left( x_1, x_2, -\frac{\xi_{o3}}{|\xi_{o3}|} \rho, t, s \right) \right| \left| \ell \partial_{x_3} u_j \left( x_1, x_2, -\frac{\xi_{o3}}{|\xi_{o3}|} \rho, t \right) \right| d\sigma .
\end{aligned} \tag{6.4.2}$$

Next, recall that

$$\begin{aligned}
& |(\mathcal{A}(x_o, \xi_{o3}, n) f)(x, t, s)| \\
& \leq \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} |\widehat{\varphi f}(\xi, \tau)| \left( \frac{\sqrt{\theta}}{(\sqrt{(hs)^2 + \theta^2})^{1/2}} e^{-\frac{\theta}{4} \frac{(t+2\tau hs)^2}{(hs)^2 + \theta^2}} \right) \\
& \quad \left( \frac{1}{\sqrt{s^2+1}} e^{-\frac{1}{4h} \frac{|(x_1-x_{o1}-2\xi_1 hs, x_2-x_{o2}-2\xi_2 hs)|^2}{s^2+1}} \right) \\
& \quad \frac{1}{(\sqrt{s^2+1})^{1/2}} \left( e^{-\frac{1}{4h} \frac{(2n\rho-2|\xi_3|hs + \frac{\xi_{o3}}{|\xi_{o3}|} [(-1)^n x_3 - x_{o3}])^2}{s^2+1}} \right) d\xi d\tau .
\end{aligned} \tag{6.4.3}$$

Here, for any  $s \in [0, L]$ ,  $h \in (0, 1]$ ,  $x_3 \in [-\rho, \rho]$ ,  $x_{o3} \in [\rho - 2r_o, \rho - r_o]$ ,  $\xi_3 \in (\xi_{o3} - 1, \xi_{o3} + 1)$ ,  $\xi_{o3} \in (2\mathbb{Z} + 1)$ , we have chosen  $(P, Q) \in \mathbb{N}^2$  (only depending on  $(\xi_{o3}, L)$ ) such that

$$s^2 + 1 \leq \left( 2n\rho - 2|\xi_3|hs + \frac{\xi_{o3}}{|\xi_{o3}|} [(-1)^n x_3 - x_{o3}] \right)^2 \quad \text{when } n \in \{-2Q, 2P+1\} . \tag{6.4.4}$$

Indeed, for any  $x_3 \in [-\rho, \rho]$  and  $x_{o3} \in [\rho - 2r_o, \rho - r_o]$

$$\begin{aligned}
\sqrt{s^2+1} & \leq \sqrt{L^2+1} \leq 4P\rho - 2(|\xi_{o3}|+1)L \quad \text{from our choice of } P \\
& \leq 4P\rho - 2|\xi_3|hs + 2\rho - |x_3 + x_{o3}| \quad \text{because } 2r_o \leq 2\rho - |x_3 + x_{o3}| \\
& \leq \left| 4P\rho - 2|\xi_3|hs + 2\rho + \frac{\xi_{o3}}{|\xi_{o3}|} [-x_3 - x_{o3}] \right|
\end{aligned} \tag{6.4.5}$$

and

$$\begin{aligned}
\sqrt{s^2+1} & \leq \sqrt{L^2+1} \leq 4Q\rho - 2(\rho - r_o) \quad \text{from our choice of } Q \\
& \leq 4Q\rho + 2|\xi_3|hs - |x_3 - x_{o3}| \quad \text{because } |x_3 - x_{o3}| \leq 2(\rho - r_o) \\
& \leq \left| -4Q\rho - 2|\xi_3|hs + \frac{\xi_{o3}}{|\xi_{o3}|} [x_3 - x_{o3}] \right| .
\end{aligned} \tag{6.4.6}$$

So (6.4.4) implies that

$$e^{-\frac{1}{4h} \frac{(2n\rho-2|\xi_3|hs + \frac{\xi_{o3}}{|\xi_{o3}|} [(-1)^n x_3 - x_{o3}])^2}{s^2+1}} \leq e^{-\frac{1}{4h}} \quad \text{when } n \in \{-2Q, 2P+1\} . \tag{6.4.7}$$

Therefore, from (6.4.3) and (6.4.7), for any  $s \in [0, L]$ ,  $h \in (0, 1]$ ,  $x_3 \in [-\rho, \rho]$ ,  $x_{o3} \in [\rho - 2r_o, \rho - r_o]$ ,  $\xi_3 \in (\xi_{o3} - 1, \xi_{o3} + 1)$ ,

$$\begin{aligned}
& |(\mathcal{A}(x_o, \xi_{o3}, n) f)(x, t, s)| \\
& \leq \frac{1}{(2\pi)^4} \frac{1}{\sqrt{s^2+1}} \frac{1}{(\sqrt{s^2+1})^{1/2}} e^{-\frac{1}{4h}} \\
& \quad \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} |\widehat{\varphi f}(\xi, \tau)| \left( \frac{\sqrt{\theta}}{(\sqrt{(hs)^2 + \theta^2})^{1/2}} e^{-\frac{\theta}{4} \frac{(t+2\tau hs)^2}{(hs)^2 + \theta^2}} \right) d\xi d\tau \quad \text{when } n \in \{-2Q, 2P+1\} .
\end{aligned} \tag{6.4.8}$$

On the other hand, by Cauchy-Schwarz inequality

$$\begin{aligned}
& \int_{-T}^T e^{-\frac{\theta}{4} \frac{(t+2\tau hs)^2}{(hs)^2 + \theta^2}} \left( \int_{\Gamma_1 \cup \Gamma_2} |\ell(x) \partial_{x_3} u_j(x, t)| d\sigma \right) dt \\
& \leq c \left( \int_{-T}^T e^{-\frac{\theta}{4} \frac{(t+2\tau hs)^2}{(hs)^2 + \theta^2}} dt \right)^{1/2} \left( \int_{-T}^T \int_{\Gamma_1 \cup \Gamma_2} e^{-\frac{\theta}{4} \frac{(t+2\tau hs)^2}{(hs)^2 + \theta^2}} |\ell(x) \partial_{x_3} u_j(x, t)|^2 d\sigma dt \right)^{1/2} \\
& \leq c \left( \frac{2\sqrt{\pi}}{\sqrt{\theta}} \sqrt{(hs)^2 + \theta^2} \right)^{1/2} \left( \int_{-T}^T \int_{\Gamma_1 \cup \Gamma_2} e^{-\frac{\theta}{4} \frac{(t+2\tau hs)^2}{(hs)^2 + \theta^2}} |\ell(x) \partial_{x_3} u_j(x, t)|^2 d\sigma dt \right)^{1/2} .
\end{aligned} \tag{6.4.9}$$

Next, by multiplying the equation  $\partial_t^2 u_j - \Delta u_j = 0$  by  $g \ell^2 \nabla u_j \cdot W$  where  $g(t) = e^{-\frac{\theta}{4} \frac{(t+2\tau hs)^2}{(hs)^2 + \theta^2}}$  and  $W = W(x)$  is a smooth vector field such that  $W = \nu$  on  $\partial\Omega$  (see [9, page 29]), we get, after integrations by parts and by Cauchy-Schwarz inequality, observing that  $\ell u_j = 0$  on  $\partial\Omega$ ,

$$\begin{aligned} \int_{\mathbb{R}} \int_{\Gamma_1 \cup \Gamma_2} g(t) |\ell(x) \partial_{x_3} u_j(x, t)|^2 d\sigma dt &\leq c \int_{\mathbb{R}} \left( g + \frac{d}{dt} g \right) \int_{\Omega} (|u_j|^2 + |\nabla u_j|^2 + |\partial_t u_j|^2) dx dt \\ &\leq c \left( \frac{2\sqrt{\pi}}{\sqrt{\theta}} \sqrt{(hs)^2 + \theta^2} \right) \mathcal{G}(U, 0) . \end{aligned} \quad (6.4.10)$$

Therefore, (6.4.9) and (6.4.10) imply that

$$\int_{-T}^T e^{-\frac{\theta}{4} \frac{(t+2\tau hs)^2}{(hs)^2 + \theta^2}} \left( \int_{\Gamma_1 \cup \Gamma_2} |\ell(x) \partial_{x_3} u_j(x, t)| d\sigma \right) dt \leq c \sqrt{(hs)^2 + \theta^2} \sqrt{\mathcal{G}(U, 0)} . \quad (6.4.11)$$

We conclude from (6.4.2), (6.4.8) and (6.4.11) that

$$\begin{aligned} &\left| h \int_{\Gamma_1 \cup \Gamma_2} \int_{-T}^T \left( \int_0^L \mathbb{A}(x_o, \xi_{o3}) f(x, t, s) ds \right) \ell(x) \partial_{\nu} u_j(x, t) d\sigma dt \right| \\ &\leq \frac{h}{(2\pi)^4} e^{-\frac{1}{4h}} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} |\widehat{\varphi f}(\xi, \tau)| d\xi d\tau \\ &\quad \int_0^L \frac{1}{\sqrt{s^2+1}} \frac{1}{(\sqrt{s^2+1})^{1/2}} \frac{\sqrt{\theta}}{(\sqrt{(hs)^2 + \theta^2})^{1/2}} \int_{-T}^T e^{-\frac{\theta}{4} \frac{(t+2\tau hs)^2}{(hs)^2 + \theta^2}} \int_{\Gamma_1 \cup \Gamma_2} |\ell(x) \partial_{x_3} u_j(x, t)| d\sigma dt ds \\ &\leq chL e^{-\frac{1}{4h}} \sqrt{\mathcal{G}(U, 0)} \left( \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} |\widehat{\varphi f}(\xi, \tau)| d\xi d\tau \right) . \end{aligned} \quad (6.4.12)$$

## 6.5 Estimate for $\mathcal{I}_4$ (the boundary term with $\partial_{x_3} \mathbb{B}$ )

We estimate  $\mathcal{I}_4 = h \int_{\Gamma_1 \cup \Gamma_2} \int_{-T}^T \left( \int_0^L \partial_{\nu} (\mathbb{B}(x_o, \xi_{o3}) f_3) ds \right) \ell u_3 d\sigma dt$  as follows.

**Lemma 6.4 .-** *There exists  $c > 0$  such that for any  $(x_o, \xi_{o3}) \in \overline{\omega_o} \times (2\mathbb{Z} + 1)$  and  $h \in (0, 1]$ ,  $L \geq 1$ ,  $\lambda \geq 1$ ,  $T > 0$ , we have*

$$|\mathcal{I}_4| \leq chL e^{-\frac{1}{4h}} \sqrt{\mathcal{G}(U, 0)} \left( \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} |\widehat{\varphi F}(\xi, \tau)| d\xi d\tau \right) . \quad (6.5.1)$$

Proof .- First, by (6.1.8), we deduce that

$$\begin{aligned} &\int_{\Gamma_1 \cup \Gamma_2} \partial_{\nu} (\mathbb{B}(x_o, \xi_{o3}) f)(x, t, s) \ell(x) u_3(x, t) d\sigma \\ &\leq \int_{\Gamma_1 \cup \Gamma_2} \left| \partial_{x_3} (\mathcal{A}(x_o, \xi_{o3}, -2Q) f) \left( x_1, x_2, -\frac{\xi_{o3}}{|\xi_{o3}|} \rho, t, s \right) \right| \left| \ell u_3 \left( x_1, x_2, -\frac{\xi_{o3}}{|\xi_{o3}|} \rho, t \right) \right| d\sigma \\ &\quad + \int_{\Gamma_1 \cup \Gamma_2} \left| \partial_{x_3} (\mathcal{A}(x_o, \xi_{o3}, 2P+1) f) \left( x_1, x_2, -\frac{\xi_{o3}}{|\xi_{o3}|} \rho, t, s \right) \right| \left| \ell u_3 \left( x_1, x_2, -\frac{\xi_{o3}}{|\xi_{o3}|} \rho, t \right) \right| d\sigma . \end{aligned} \quad (6.5.2)$$

Next, recall that

$$\begin{aligned} &|\partial_{x_3} (\mathcal{A}(x_o, \xi_{o3}, n) f)(x, t, s)| \\ &\leq \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} |\widehat{\varphi f}(\xi, \tau)| \left( \frac{\sqrt{\theta}}{(\sqrt{(hs)^2 + \theta^2})^{1/2}} e^{-\frac{\theta}{4} \frac{(t+2\tau hs)^2}{(hs)^2 + \theta^2}} \right) \\ &\quad \left( \frac{1}{\sqrt{s^2+1}} e^{-\frac{1}{4h} \frac{|(x_1-x_{o1}-2\xi_1 hs, x_2-x_{o2}-2\xi_2 hs)|^2}{s^2+1}} \right) \\ &\quad \frac{1}{(\sqrt{s^2+1})^{1/2}} \left[ |\xi_3| + \frac{1}{\sqrt{h}} \right] \left( e^{-\frac{1}{8h} \frac{(2n\rho - 2|\xi_3|hs + \frac{\xi_{o3}}{|\xi_{o3}|} [(-1)^n x_3 - x_{o3}])^2}{s^2+1}} \right) d\xi d\tau . \end{aligned} \quad (6.5.3)$$

Here, for any  $s \in [0, L]$ ,  $h \in (0, 1]$ ,  $x_3 \in [-\rho, \rho]$ ,  $x_{o3} \in [\rho - 2r_o, \rho - r_o]$ ,  $\xi_3 \in (\xi_{o3} - 1, \xi_{o3} + 1)$ ,  $\xi_{o3} \in (2\mathbb{Z} + 1)$ , we have chosen  $(P, Q) \in \mathbb{N}^2$  (only depending on  $(\xi_{o3}, L)$ ) such that

$$(h|\xi_3| + 1)(s^2 + 1) \leq \left( 2n\rho - 2|\xi_3|hs + \frac{\xi_{o3}}{|\xi_{o3}|} [(-1)^n x_3 - x_{o3}] \right)^2 \quad \text{when } n \in \{-2Q, 2P + 1\}. \quad (6.5.4)$$

Indeed, for any  $x_3 \in [-\rho, \rho]$  and  $x_{o3} \in [\rho - 2r_o, \rho - r_o]$

$$\begin{aligned} \sqrt{(h|\xi_3| + 1)(s^2 + 1)} &\leq \sqrt{(|\xi_{o3}| + 2)(L^2 + 1)} \leq 4P\rho - 2(|\xi_{o3}| + 1)L \\ &\leq 4P\rho - 2|\xi_3|hs + 2\rho - |x_3 + x_{o3}| \quad \text{because } 0 \leq 2\rho - |x_3 + x_{o3}| \\ &\leq \left| 4P\rho - 2|\xi_3|hs + 2\rho + \frac{\xi_{o3}}{|\xi_{o3}|} [-x_3 - x_{o3}] \right| \end{aligned} \quad (6.5.5)$$

and

$$\begin{aligned} \sqrt{(h|\xi_3| + 1)(s^2 + 1)} &\leq \sqrt{(|\xi_{o3}| + 2)(L^2 + 1)} \leq 4Q\rho - 2(\rho - r_o) \\ &\leq 4Q\rho + 2|\xi_3|hs - |x_3 - x_{o3}| \quad \text{because } |x_3 - x_{o3}| \leq 2(\rho - r_o) \\ &\leq \left| -4Q\rho - 2|\xi_3|hs + \frac{\xi_{o3}}{|\xi_{o3}|} [x_3 - x_{o3}] \right|. \end{aligned} \quad (6.5.6)$$

So (6.5.4) implies that when  $n \in \{-2Q, 2P + 1\}$

$$\left[ |\xi_3| + \frac{1}{\sqrt{h}} \right] e^{-\frac{1}{8h} \frac{(2n\rho - 2|\xi_3|hs + \frac{\xi_{o3}}{|\xi_{o3}|} [(-1)^n x_3 - x_{o3}])^2}{s^2 + 1}} \leq \left[ |\xi_3| + \frac{1}{h} \right] e^{-\frac{1}{8}(|\xi_3| + \frac{1}{h})} \leq 16e^{-\frac{1}{16h}}. \quad (6.5.7)$$

Therefore, from (6.5.3) and (6.5.7), for any  $s \in [0, L]$ ,  $h \in (0, 1]$ ,  $x_3 \in [-\rho, \rho]$ ,  $x_{o3} \in [\rho - 2r_o, \rho - r_o]$ ,  $\xi_3 \in (\xi_{o3} - 1, \xi_{o3} + 1)$ ,

$$\begin{aligned} &|\partial_{x_3}(\mathcal{A}(x_o, \xi_{o3}, n)f)(x, t, s)| \\ &\leq \frac{1}{(2\pi)^4} \frac{1}{\sqrt{s^2 + 1}} \frac{1}{(\sqrt{s^2 + 1})^{1/2}} c e^{-\frac{1}{ch}} \\ &\quad \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} |\widehat{\varphi f}(\xi, \tau)| \left( \frac{\sqrt{\theta}}{(\sqrt{(hs)^2 + \theta^2})^{1/2}} e^{-\frac{\theta}{4} \frac{(t+2\tau hs)^2}{(hs)^2 + \theta^2}} \right) d\xi d\tau \quad \text{when } n \in \{-2Q, 2P + 1\}. \end{aligned} \quad (6.5.8)$$

On the other hand, by Cauchy-Schwarz inequality, a trace theorem and (6.1.12), we have

$$\begin{aligned} &\int_{-T}^T e^{-\frac{\theta}{4} \frac{(t+2\tau hs)^2}{(hs)^2 + \theta^2}} \left( \int_{\Gamma_1 \cup \Gamma_2} |\ell(x) u_3(x, t)| d\sigma \right) dt \\ &\leq c \left( \int_{-T}^T e^{-\frac{\theta}{4} \frac{(t+2\tau hs)^2}{(hs)^2 + \theta^2}} dt \right)^{1/2} \left( \int_{-T}^T e^{-\frac{\theta}{4} \frac{(t+2\tau hs)^2}{(hs)^2 + \theta^2}} \int_{\Gamma_1 \cup \Gamma_2} |u_3(x, t)|^2 d\sigma dt \right)^{1/2} \\ &\leq c \left( \frac{2\sqrt{\pi}}{\sqrt{\theta}} \sqrt{(hs)^2 + \theta^2} \right)^{1/2} \left( \int_{-T}^T e^{-\frac{\theta}{4} \frac{(t+2\tau hs)^2}{(hs)^2 + \theta^2}} \|u_3(\cdot, t)\|_{H^1(\Omega)}^2 dt \right)^{1/2} \\ &\leq c \left( \sqrt{(hs)^2 + \theta^2} \right) \sqrt{\mathcal{G}(U, 0)}. \end{aligned} \quad (6.5.9)$$

We conclude from (6.5.2), (6.5.8) and (6.5.9) that

$$\begin{aligned} &\left| h \int_{\Gamma_1 \cup \Gamma_2} \int_{-T}^T \left( \int_0^L \partial_\nu(\mathbb{B}(x_o, \xi_{o3})f)(x, t, s) ds \right) \ell(x) u_3(x, t) d\sigma dt \right| \\ &\leq \frac{h}{(2\pi)^4} c e^{-\frac{1}{ch}} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} |\widehat{\varphi f}(\xi, \tau)| d\xi d\tau \\ &\quad \int_0^L \frac{1}{\sqrt{s^2 + 1}} \frac{1}{(\sqrt{s^2 + 1})^{1/2}} \frac{\sqrt{\theta}}{(\sqrt{(hs)^2 + \theta^2})^{1/2}} \int_{-T}^T e^{-\frac{\theta}{4} \frac{(t+2\tau hs)^2}{(hs)^2 + \theta^2}} \int_{\Gamma_1 \cup \Gamma_2} |\ell(x) u_3(x, t)| d\sigma dt ds \\ &\leq chL e^{-\frac{1}{ch}} \sqrt{\mathcal{G}(U, 0)} \left( \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} |\widehat{\varphi f}(\xi, \tau)| d\xi d\tau \right). \end{aligned} \quad (6.5.10)$$

## 6.6 Estimate for $\mathcal{I}_5$ (the boundary term on $(\Gamma_1 \cup \Gamma_2) \cap \Theta$ )

We estimate  $\mathcal{I}_5 = -h \int_{(\Gamma_1 \cup \Gamma_2) \cap \Theta} \int_{-T}^T \left( \int_0^L \mathbb{B}(x_o, \xi_{o3}) f_3 ds \right) \partial_\nu \ell u_3 d\sigma dt$  as follows.

**Lemma 6.5 .-** *There exists  $c > 0$  such that for any  $(x_o, \xi_{o3}) \in \overline{\omega_o} \times (2\mathbb{Z} + 1)$  and  $h \in (0, 1]$ ,  $L \geq 1$ ,  $\lambda \geq 1$ ,  $T > 0$ , we have*

$$|\mathcal{I}_5| \leq ch \left( 1 + \sqrt{hL} \right) \| (u_3, \partial_t u_3) \|_{L^2(\omega \times (-1-T, T+1))}^2 \left( \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} \left| \widehat{\varphi f}(\xi, \tau) \right| d\xi d\tau \right). \quad (6.6.1)$$

Proof .- Since

$$\begin{aligned} |(\mathbb{B}(x_o, \xi_{o3}) f)(x, t, s)| &\leq \sum_{n \in \mathbb{Z}} |(\mathcal{A}(x_o, \xi_{o3}, n) f)(x, t, s)| \\ &\leq \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} \left| \widehat{\varphi f}(\xi, \tau) \right| \\ &\quad \left( \frac{1}{\sqrt{s^2+1}} e^{-\frac{1}{4h} \frac{|(x_1-x_{o1}-2\xi_1 h s, x_2-x_{o2}-2\xi_2 h s)|^2}{s^2+1}} \right) \left( \frac{\sqrt{\theta}}{(\sqrt{(hs)^2+\theta^2})^{1/2}} e^{-\frac{\theta}{4} \frac{(t+2\tau h s)^2}{(hs)^2+\theta^2}} \right) \\ &\quad \left( \frac{1}{(\sqrt{s^2+1})^{1/2}} \sum_{n \in \mathbb{Z}} e^{-\frac{1}{4h} \frac{((-1)^n x_3 + 2n \frac{\xi_{o3}}{|\xi_{o3}|} \rho - x_{o3} - 2\xi_3 h s)^2}{s^2+1}} \right) d\xi d\tau \end{aligned} \quad (6.6.2)$$

and

$$\sum_{n \in \mathbb{Z}} e^{-\frac{1}{4h} \frac{((-1)^n x_3 + 2n \frac{\xi_{o3}}{|\xi_{o3}|} \rho - x_{o3} - 2\xi_3 h s)^2}{s^2+1}} \leq 2 + \frac{\sqrt{\pi}}{\rho} \sqrt{h} \sqrt{s^2+1} \quad (6.6.3)$$

(see Appendix B with  $z = \frac{4h}{\rho^2} (s^2+1)$ ), we have

$$\begin{aligned} &\left| h \int_{(\Gamma_1 \cup \Gamma_2) \cap \Theta} \int_{-T}^T \left( \int_0^L \mathbb{B}(x_o, \xi_{o3}) f ds \right) \partial_\nu \ell u_3 d\sigma dt \right| \\ &\leq h \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} \left| \widehat{\varphi f}(\xi, \tau) \right| \int_0^L \frac{1}{\sqrt{s^2+1}} \frac{(2 + \frac{\sqrt{\pi}}{\rho} \sqrt{h} \sqrt{s^2+1})}{(\sqrt{s^2+1})^{1/2}} \\ &\quad \left( \frac{\sqrt{\theta}}{(\sqrt{(hs)^2+\theta^2})^{1/2}} \int_{-T}^T e^{-\frac{\theta}{4} \frac{(t+2\tau h s)^2}{(hs)^2+\theta^2}} \int_{(\Gamma_1 \cup \Gamma_2) \cap \Theta} |\partial_\nu \ell u_3(\cdot, t)| d\sigma dt \right) ds d\xi d\tau \\ &\leq ch \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} \left| \widehat{\varphi f}(\xi, \tau) \right| d\xi d\tau \\ &\quad \int_0^L \frac{1}{\sqrt{s^2+1}} \frac{(1 + \sqrt{h} \sqrt{s^2+1})}{(\sqrt{s^2+1})^{1/2}} \frac{\sqrt{\theta}}{(\sqrt{(hs)^2+\theta^2})^{1/2}} \left( \int_{-\infty}^{\infty} e^{-\frac{\theta}{2} \frac{t^2}{(hs)^2+\theta^2}} dt \right)^{1/2} ds \\ &\quad \left( \int_{-T}^T \int_{(\Gamma_1 \cup \Gamma_2) \cap \Theta} |\partial_\nu \ell u_3|^2 d\sigma dt \right)^{1/2}. \end{aligned} \quad (6.6.4)$$

Now, we shall treat the term  $\left( \int_{-T}^T \int_{(\Gamma_1 \cup \Gamma_2) \cap \Theta} |\partial_\nu \ell u_3|^2 d\sigma dt \right)^{1/2}$  as follows. Let  $W = W(x)$  be a smooth vector field such that  $W = \nu$  on  $\partial\Omega$  (see [9, page 29]). Since

$$\operatorname{div} \left( W u^2 (\nabla \ell \cdot W)^2 \right) = 2u (\nabla u \cdot W) (\nabla \ell \cdot W)^2 + u^2 \nabla \left[ (\nabla \ell \cdot W)^2 \right] \cdot W + u^2 (\nabla \ell \cdot W)^2 \operatorname{div} W, \quad (6.6.5)$$

we have the following trace theorem

$$\int_{\partial\Omega} |\partial_\nu \ell u|^2 d\sigma \leq c \int_{\omega} |u|^2 dx + c \int_{\omega} |\nabla u|^2 (\nabla \ell \cdot W)^2 dx. \quad (6.6.6)$$



Next, by multiplying the equation  $\partial_t^2 u_3 - \Delta u_3 = 0$  by  $u_3 (\nabla \ell \cdot W)^2 g$  where  $g \in C_0^\infty(-1-T, T+1)$  and  $g = 1$  in  $(-T, T)$ , we get, after integrations by parts and by Cauchy-Schwarz inequality, observing that  $\partial_\nu u_3 \partial_\nu \ell = 0$  on  $\partial\Omega$ ,

$$\int_{\Omega} \int_{-T}^T |\nabla u_3|^2 (\nabla \ell \cdot W)^2 dx dt \leq c \int_{\omega} \int_{-1-T}^{T+1} (|u_3|^2 + |\partial_t u_3|^2) dx dt. \quad (6.6.7)$$

Therefore, combining (6.6.6), (6.6.7) and (6.6.4), we conclude that

$$\begin{aligned} & \left| h \int_{(\Gamma_1 \cup \Gamma_2) \cap \Theta} \int_{-T}^T \left( \int_0^L \mathbb{B}(x_o, \xi_{o3}) f ds \right) \partial_\nu \ell u_3 d\sigma dt \right| \\ & \leq ch \left( 1 + \sqrt{hL} \right) \| (u_3, \partial_t u_3) \|_{L^2(\omega \times (-1-T, T+1))^2} \left( \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} |\widehat{\varphi f}(\xi, \tau)| d\xi d\tau \right). \end{aligned} \quad (6.6.8)$$

## 6.7 Estimate for $\mathcal{I}_6$ (the term at $t = \pm T$ )

We estimate  $\mathcal{I}_6 = h \int_{\Omega} \left[ \left( \int_0^L \partial_t (V(x_o, \xi_{o3}) F)(\cdot, t, \cdot) ds \right) \cdot \ell U(\cdot, t) - \left( \int_0^L (V(x_o, \xi_{o3}) F)(\cdot, t, \cdot) ds \right) \cdot \ell \partial_t U(\cdot, t) \right]_{-T}^T dx$  as follows.

**Lemma 6.6 .-** *There exists  $c > 0$  such that for any  $(x_o, \xi_{o3}) \in \overline{\omega_o} \times (2\mathbb{Z} + 1)$  and  $h \in (0, 1]$ ,  $L \geq 1$ ,  $\lambda \geq 1$ , we have*

$$|\mathcal{I}_6| \leq chL\lambda e^{-\frac{1}{h}} \sqrt{\mathcal{G}(U, 0)} \left( \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} |\widehat{\varphi f}(\xi, \tau)| d\xi d\tau \right) \quad (6.7.1)$$

when

$$T = 4 \left[ \frac{\lambda h L}{\sqrt{2}} + \sqrt{h} L + \frac{\sqrt{2}}{\sqrt{h}} \right]. \quad (6.7.2)$$

Proof .- Since

$$\begin{aligned} & |\partial_t (\mathcal{A}(x_o, \xi_{o3}, n) f)(x, t, s)| \\ & \leq \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} |\widehat{\varphi f}(\xi, \tau)| \\ & \quad \left( \frac{1}{\sqrt{s^2+1}} e^{-\frac{1}{4h} \frac{|(x_1-x_{o1}-2\xi_1 h s, x_2-x_{o2}-2\xi_2 h s)|^2}{s^2+1}} \right) \left( \frac{1}{(\sqrt{s^2+1})^{1/2}} e^{-\frac{1}{4h} \frac{((-1)^n x_3 + 2n \frac{\xi_{o3}}{|\xi_{o3}|} \rho - x_{o3} - 2\xi_3 h s)^2}{s^2+1}} \right) \\ & \quad \left[ |\tau| + \frac{1}{2} \frac{|t+2\tau h s|}{\sqrt{(h s)^2 + \theta^2}} \right] \left( \frac{\sqrt{\theta}}{(\sqrt{(h s)^2 + \theta^2})^{1/2}} e^{-\frac{\theta}{4} \frac{(t+2\tau h s)^2}{(h s)^2 + \theta^2}} \right) d\xi d\tau, \end{aligned} \quad (6.7.3)$$

we have

$$\begin{aligned} & |(\mathbb{A}(x_o, \xi_{o3}) f)(x, \pm T, s)| + |\partial_t (\mathbb{A}(x_o, \xi_{o3}) f)(x, \pm T, s)| \\ & + |(\mathbb{B}(x_o, \xi_{o3}) f)(x, \pm T, s)| + |\partial_t (\mathbb{B}(x_o, \xi_{o3}) f)(x, \pm T, s)| \\ & \leq c \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} |\widehat{\varphi f}(\xi, \tau)| \frac{1}{\sqrt{s^2+1}} \left( \frac{1}{(\sqrt{s^2+1})^{1/2}} \sum_{n \in \mathbb{Z}} e^{-\frac{1}{4h} \frac{((-1)^n x_3 + 2n \frac{\xi_{o3}}{|\xi_{o3}|} \rho - x_{o3} - 2\xi_3 h s)^2}{s^2+1}} \right) \\ & \quad \left( 1 + \lambda + \frac{\sqrt{\theta}}{2} \frac{|t+2\tau h s|}{\sqrt{(h s)^2 + \theta^2}} \right) \left( \frac{\sqrt{\theta}}{(\sqrt{(h s)^2 + \theta^2})^{1/2}} e^{-\frac{\theta}{4} \frac{(\pm T + 2\tau h s)^2}{(h s)^2 + \theta^2}} \right) d\xi d\tau \\ & \leq c \left( \frac{1}{\sqrt{s^2+1}} \frac{1 + \sqrt{h} \sqrt{s^2+1}}{(\sqrt{s^2+1})^{1/2}} \frac{1}{(\sqrt{(h s)^2 + \theta^2})^{1/2}} \right) \lambda \left( \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} |\widehat{\varphi f}(\xi, \tau)| e^{-\frac{\theta}{8} \frac{(\pm T + 2\tau h s)^2}{(h s)^2 + \theta^2}} d\xi d\tau \right). \end{aligned} \quad (6.7.4)$$

On the other hand,

$$e^{-\frac{\theta}{8} \frac{(\pm T + 2\tau h s)^2}{(hs)^2 + \theta^2}} \leq e^{-\frac{\theta T^2}{16} \frac{1}{(hL)^2 + \theta^2}} e^{\frac{\theta}{8} \frac{(2\lambda hL)^2}{(hL)^2 + \theta^2}} \quad \forall s \in [0, L], |\tau| < \lambda. \quad (6.7.5)$$

Now, when  $T = 4 \left[ \frac{\lambda hL}{\sqrt{2}} + \sqrt{h}L + \frac{\sqrt{2}}{\sqrt{h}} \right]$ , then  $\frac{1}{2} (\lambda hL)^2 + \frac{1}{\theta h} \left( (hL)^2 + \theta^2 \right) \leq \frac{T^2}{16}$  which implies that

$$e^{-\frac{\theta T^2}{16} \frac{1}{(hL)^2 + \theta^2}} e^{\frac{\theta}{8} \frac{(2\lambda hL)^2}{(hL)^2 + \theta^2}} \leq e^{-\frac{1}{h}}. \quad (6.7.6)$$

In conclusion, combining (6.7.4), (6.7.5), (6.7.6) and (6.1.12), we get

$$\begin{aligned} & \left| h \int_{\Omega} \left[ \left( \int_0^L \partial_t (V(x_o, \xi_{o3}) F)(\cdot, t, \cdot) ds \right) \cdot \ell U(\cdot, t) - \left( \int_0^L (V(x_o, \xi_{o3}) F)(\cdot, t, \cdot) ds \right) \cdot \ell \partial_t U(\cdot, t) \right]_{-T}^T dx \right| \\ & \leq chL\lambda e^{-\frac{1}{h}} \sqrt{\mathcal{G}(U, 0)} \left( \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} |\widehat{\varphi F}(\xi, \tau)| d\xi d\tau \right). \end{aligned} \quad (6.7.7)$$

## 6.8 Estimate for $\mathcal{I}_7$ (the internal term in $\omega$ )

We estimate  $\mathcal{I}_7 = h \int_{\omega} \int_{-T}^T \left( \int_0^L V(x_o, \xi_{o3}) F ds \right) \cdot [2(\nabla \ell \cdot \nabla) U + \Delta \ell U] dx dt$  as follows.

**Lemma 6.7 .-** *There exists  $c > 0$  such that for any  $(x_o, \xi_{o3}) \in \overline{\omega_o} \times (2\mathbb{Z} + 1)$  and  $h \in (0, 1]$ ,  $L \geq 1$ ,  $\lambda \geq 1$ ,  $T > 0$ , we have*

$$|\mathcal{I}_7| \leq ch \left( 1 + \sqrt{hL} \right) \left( \|(U, \partial_t U)\|_{L^2(\omega \times (-1-T, T+1))}^6 \right) \left( \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} |\widehat{\varphi F}(\xi, \tau)| d\xi d\tau \right). \quad (6.8.1)$$

Proof .- We start with the third component of  $V(x_o, \xi_{o3}) F$ . Since

$$\begin{aligned} & |(\mathbb{B}(x_o, \xi_{o3}) f)(x, t, s)| \leq \sum_{n \in \mathbb{Z}} |(\mathcal{A}(x_o, \xi_{o3}, n) f)(x, t, s)| \\ & \leq \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} |\widehat{\varphi f}(\xi, \tau)| \left( \frac{\sqrt{\theta}}{(\sqrt{(hs)^2 + \theta^2})^{1/2}} e^{-\frac{\theta}{4} \frac{(t+2\tau h s)^2}{(hs)^2 + \theta^2}} \right) d\xi d\tau \\ & \quad \left( \frac{1}{\sqrt{s^2+1}} \right) \left( \frac{2 + \frac{\sqrt{\pi}}{\rho} \sqrt{h} \sqrt{s^2+1}}{(\sqrt{s^2+1})^{1/2}} \right) \end{aligned} \quad (6.8.2)$$

(see (6.6.2)-(6.6.3)), we have

$$\begin{aligned} & \left| h \int_{\omega} \int_{-T}^T \left( \int_0^L (\mathbb{B}(x_o, \xi_{o3}) f)(x, t, s) ds \right) [2\nabla \ell \nabla u_3 + \Delta \ell u_3](x, t) dx dt \right| \\ & \leq ch \int_{\omega} \int_{-T}^T \left( \int_0^L |(\mathbb{B}(x_o, \xi_{o3}) f)(x, t, s)| ds \right) (|\nabla \ell| |\nabla u_3| + |u_3|)(x, t) dx dt \\ & \leq ch \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} |\widehat{\varphi f}(\xi, \tau)| \int_0^L \frac{1}{\sqrt{s^2+1}} \frac{(2 + \frac{\sqrt{\pi}}{\rho} \sqrt{h} \sqrt{s^2+1})}{(\sqrt{s^2+1})^{1/2}} \\ & \quad \left( \int_{-T}^T \left( \frac{\sqrt{\theta}}{(\sqrt{(hs)^2 + \theta^2})^{1/2}} e^{-\frac{\theta}{4} \frac{(t+2\tau h s)^2}{(hs)^2 + \theta^2}} \right) \int_{\omega} (|\nabla \ell| |\nabla u_3| + |u_3|)(x, t) dx dt \right) ds d\xi d\tau \end{aligned} \quad (6.8.3)$$

which implies using Cauchy-Schwarz inequality

$$\begin{aligned}
& \left| h \int_{\omega} \int_{-T}^T \left( \int_0^L (\mathbb{B}(x_o, \xi_{o3}) f)(x, t, s) ds \right) [2\nabla \ell \nabla u_3 + \Delta \ell u_3](x, t) dx dt \right| \\
& \leq ch \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} |\widehat{\varphi f}(\xi, \tau)| d\xi d\tau \\
& \quad \int_0^L \frac{1}{\sqrt{s^2+1}} \frac{(1+\sqrt{h}\sqrt{s^2+1})}{(\sqrt{s^2+1})^{1/2}} \left( \frac{\sqrt{\theta}}{(\sqrt{(hs)^2+\theta^2})^{1/2}} \right) \left( \int_{-\infty}^{\infty} e^{-\frac{\theta}{2} \frac{t^2}{(hs)^2+\theta^2}} dt \right)^{1/2} ds \\
& \quad \left( \int_{\omega} \int_{-T}^T (|\nabla \ell|^2 |\nabla u_3|^2 + |u_3|^2) dx dt \right)^{1/2} \\
& \leq ch (1 + \sqrt{hL}) \left( \int_{\omega} \int_{-T}^T (|\nabla \ell|^2 |\nabla u_3|^2 + |u_3|^2) dx dt \right)^{1/2} \left( \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} |\widehat{\varphi f}(\xi, \tau)| d\xi d\tau \right)^{1/2}.
\end{aligned} \tag{6.8.4}$$

Similarly, for any  $j \in \{1, 2\}$ ,

$$\begin{aligned}
& \left| h \int_{\omega} \int_{-T}^T \left( \int_0^L (\mathbb{A}(x_o, \xi_{o3}) f)(x, t, s) ds \right) [2\nabla \ell \nabla u_j + \Delta \ell u_j](x, t) dx dt \right| \\
& \leq ch (1 + \sqrt{hL}) \left( \int_{\omega} \int_{-T}^T (|\nabla \ell|^2 |\nabla u_j|^2 + |u_j|^2) dx dt \right)^{1/2} \left( \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} |\widehat{\varphi f}(\xi, \tau)| d\xi d\tau \right)^{1/2}.
\end{aligned} \tag{6.8.5}$$

Now, we shall bound the term  $\left( \int_{\omega} \int_{-T}^T (|\nabla \ell|^2 |\nabla u_j|^2 + |u_j|^2) dx dt \right)^{1/2}$  for any  $j \in \{1, 2, 3\}$  by the quantity  $\|(U, \partial_t U)\|_{L^2(\omega \times (-1-T, T+1))}^6$ . By multiplying the equation  $\partial_t^2 u_j - \Delta u_j = 0$  by  $u_j |\nabla \ell|^2 g$  where  $g \in C_0^\infty(-1-T, T+1)$  and  $g = 1$  in  $(-T, T)$ , we get, after integrations by parts and by Cauchy-Schwarz inequality, observing that  $u_j \partial_\nu u_j |\nabla \ell| = 0$  on  $\partial\Omega$ ,

$$\int_{\Omega} \int_{-T}^T |\nabla u_j|^2 |\nabla \ell|^2 dx dt \leq c \int_{\omega} \int_{-1-T}^{T+1} (|u_j|^2 + |\partial_t u_j|^2) dx dt, \tag{6.8.6}$$

for any  $j \in \{1, 2, 3\}$ . This completes the proof.

## 6.9 Key inequality

From now,

$$T = 4 \left[ \frac{\lambda h L}{\sqrt{2}} + \sqrt{h} L + \frac{\sqrt{2}}{\sqrt{h}} \right]. \tag{6.9.1}$$

By (6.1.13), (6.2.1), (6.3.1), (6.4.1), (6.5.1), (6.6.1), (6.7.1) and (6.8.1), there exists  $c > 0$  such that for any  $(x_o, \xi_{o3}) \in \overline{\omega_o} \times (2\mathbb{Z} + 1)$  and  $h \in (0, 1]$ ,  $L \geq 1$ ,  $\lambda \geq 1$ , we have

$$\begin{aligned}
& \left| \int_{\Omega \times \mathbb{R}} \left( \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} e^{i(x\xi+t\tau)} \widehat{\varphi F}(\xi, \tau) d\xi d\tau \right) \cdot a_{o,\theta}(x, t) \ell(x) U(x, t) dx dt \right| \\
& \leq c \left[ (1 + hL\lambda) e^{-\frac{1}{ch}} + \frac{1}{\sqrt{L}} \right] \sqrt{\mathcal{G}(U, 0)} \left( \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} |\widehat{\varphi F}(\xi, \tau)| d\xi d\tau \right) \\
& \quad + ch (1 + \sqrt{hL}) \left( \|(U, \partial_t U)\|_{L^2(\omega \times (-1-T, T+1))}^6 \right) \left( \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau| < \lambda} |\widehat{\varphi F}(\xi, \tau)| d\xi d\tau \right).
\end{aligned} \tag{6.9.2}$$

By summing over  $\xi_{o3} \in (2\mathbb{Z} + 1)$ , it implies that

$$\begin{aligned} & \left| \int_{\Omega \times \mathbb{R}} \left( \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{|\tau| < \lambda} e^{i(x\xi + t\tau)} \widehat{\varphi F}(\xi, \tau) d\xi d\tau \right) \cdot a_{o,\theta}(x, t) \ell(x) U(x, t) dx dt \right| \\ & \leq c \left[ (1 + hL\lambda) e^{-\frac{1}{ch}} + \frac{1}{\sqrt{L}} \right] \sqrt{\mathcal{G}(U, 0)} \left( \int_{\mathbb{R}^3} \int_{|\tau| < \lambda} |\widehat{\varphi F}(\xi, \tau)| d\xi d\tau \right) \\ & \quad + ch \left( 1 + \sqrt{hL} \right) \left( \| (U, \partial_t U) \|_{L^2(\omega \times (-1-T, T+1))^6} \right) \left( \int_{\mathbb{R}^3} \int_{|\tau| < \lambda} |\widehat{\varphi F}(\xi, \tau)| d\xi d\tau \right). \end{aligned} \quad (6.9.3)$$

On the other hand, from (A2) of Appendix A,

$$\int_{\mathbb{R}^3} \int_{|\tau| < \lambda} |\widehat{\varphi F}(\xi, \tau)| d\xi d\tau \leq c\sqrt{\lambda} \left( \frac{\lambda^2}{\sqrt{h}} + \frac{1}{h} \right) \sqrt{\mathcal{G}(\partial_t U, 0)} \quad (6.9.4)$$

whenever  $\varphi F = \varphi_2 U$  or  $\varphi F = \varphi_1 \partial_t^2 U$ . Therefore, by (6.9.3) with (6.9.4), we obtain that

$$\begin{aligned} & 2 \left| \int_{\Omega \times \mathbb{R}} \left( \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{|\tau| < \lambda} e^{i(x\xi + t\tau)} \widehat{\varphi_2 U}(\xi, \tau) d\xi d\tau \right) \cdot a_{o,2}(x, t) \ell(x) U(x, t) dx dt \right| \\ & + \left| \int_{\Omega \times \mathbb{R}} \left( \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{|\tau| < \lambda} e^{i(x\xi + t\tau)} \widehat{\varphi_1 \partial_t^2 U}(\xi, \tau) d\xi d\tau \right) \cdot a_{o,1}(x, t) \ell(x) U(x, t) dx dt \right| \\ & \leq c \left[ (1 + hL\lambda) e^{-\frac{1}{ch}} + \frac{1}{\sqrt{L}} \right] \sqrt{\lambda} \left( \frac{\lambda^2}{\sqrt{h}} + \frac{1}{h} \right) \mathcal{G}(\partial_t U, 0) \\ & \quad + ch \left( 1 + \sqrt{hL} \right) \left( \| (U, \partial_t U) \|_{L^2(\omega \times (-1-T, T+1))^6} \right) \sqrt{\lambda} \left( \frac{\lambda^2}{\sqrt{h}} + \frac{1}{h} \right) \sqrt{\mathcal{G}(\partial_t U, 0)}, \end{aligned} \quad (6.9.5)$$

which is our claim (6.8). This completes the proof.

## Appendix A

The goal of this Appendix A is to prove the two following inequalities (A1) and (A2) below.

**Lemma A .-** *Let*

$$a_o(x, t) = e^{-\frac{1}{c_1 h} |x - x_o|^2} e^{-\frac{1}{c_2} t^2} \quad \text{and} \quad \varphi(x, t) = \phi(x) e^{-\frac{1}{c_3 h} |x - x_o|^2} e^{-\frac{1}{c_4} t^2}$$

for some  $c_1, c_2, c_3, c_4 > 0$  and  $\phi \in C_0^\infty(\Omega)$ . Let  $\ell \in C^\infty(\mathbb{R}^3)$  be such that  $0 \leq \ell(x) \leq 1$ . There exists  $c > 0$  such that for any  $h \in (0, 1]$ ,  $\lambda \geq 1$  and any  $u \in C^1(\mathbb{R}, H^1(\Omega)) \cap C^2(\mathbb{R}, L^2(\Omega))$  satisfying

$$\partial_t^2 u - \Delta u = 0 \quad \text{in } \Omega \times \mathbb{R},$$

and

$$\exists R_j > 0 \quad \left\| \partial_t^j u(\cdot, t) \right\|_{L^2(\Omega)}^2 \leq R_j \quad \text{for } j \in \{0, 1, 2\}, \quad \|\nabla u(\cdot, t)\|_{L^2(\Omega)}^2 \leq R_1,$$

we have

$$\begin{aligned} & \left| \int_{\Omega \times \mathbb{R}} \left( \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{|\tau| \geq \lambda} e^{i(x\xi + t\tau)} \widehat{\varphi f}(\xi, \tau) d\xi d\tau \right) a_o(x, t) \ell(x) u(x, t) dx dt \right| \\ & \leq c\sqrt{\frac{1}{\lambda}} (R_0 + R_1) (R_0 + R_2) \end{aligned} \quad (A1)$$

and

$$\left( \int_{\mathbb{R}^3} \int_{|\tau| < \lambda} |\widehat{\varphi f}(\xi, \tau)| d\xi d\tau \right) \leq c\sqrt{\lambda} \left( \left( \lambda^2 + \frac{1}{h} \right) R_2 + \frac{\lambda^2}{\sqrt{h}} R_1 + \frac{1}{h} R_0 \right) \quad (A2)$$

whenever  $f = u$  or  $f = \partial_t^2 u$ .

**Proof of (A1).** Introduce

$$\mathcal{R}(f) = \int_{\Omega \times \mathbb{R}} \left( \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{|\tau| \geq \lambda} e^{i(x\xi + t\tau)} \widehat{\varphi f}(\xi, \tau) d\xi d\tau \right) a_o(x, t) \ell(x) u(x, t) dx dt .$$

Thus,

$$\begin{aligned} |\mathcal{R}(f)| &= \left| \int_{\Omega \times \mathbb{R}} a_o(x, t) \ell(x) u(x, t) \partial_t \left( \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{|\tau| \geq \lambda} \frac{1}{i\tau} e^{i(x\xi + t\tau)} \widehat{\varphi f}(\xi, \tau) d\xi d\tau \right) dx dt \right| , \\ &= \left| \int_{\Omega \times \mathbb{R}} \ell(x) \partial_t(a_o u(x, t)) \left( \frac{1}{2\pi} \int_{|\tau| \geq \lambda} \frac{1}{i\tau} e^{it\tau} \left[ \int_{\mathbb{R}} e^{-i\theta\tau} (\varphi f)(x, \theta) d\theta \right] d\tau \right) dx dt \right| . \end{aligned}$$

It follows using Cauchy-Schwarz inequality and Parseval identity that

$$\begin{aligned} |\mathcal{R}(f)| &\leq \int_{\Omega \times \mathbb{R}} |\ell(x) \partial_t(a_o u(x, t))| \left( \frac{1}{2\pi} \left[ \int_{|\tau| \geq \lambda} \frac{1}{\tau^2} d\tau \right]^{1/2} \left[ \int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{-i\theta\tau} (\varphi f)(x, \theta) d\theta \right|^2 d\tau \right]^{1/2} \right) dx dt \\ &\leq \int_{\Omega \times \mathbb{R}} |\partial_t(a_o u(x, t))| \left( \frac{1}{2\pi} \left[ \int_{|\tau| \geq \lambda} \frac{1}{\tau^2} d\tau \right]^{1/2} \left[ 2\pi \int_{\mathbb{R}} |(\varphi f)(x, \theta)|^2 d\theta \right]^{1/2} \right) dx dt \\ &\leq \int_{\Omega \times \mathbb{R}} |\partial_t(a_o u(x, t))| \left( \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\lambda}} \|(\varphi f)(x, \cdot)\|_{L^2(\mathbb{R})} \right) dx dt \\ &\leq \frac{1}{\sqrt{\pi}} \sqrt{\frac{1}{\lambda}} \int_{\mathbb{R}} \|\partial_t(a_o u)(\cdot, t)\|_{L^2(\Omega)} dt \|\varphi f\|_{L^2(\Omega \times \mathbb{R})} . \end{aligned}$$

Since we have the following estimates

$$\begin{aligned} \int_{\mathbb{R}} e^{-\frac{t^2}{c}} \int_{\Omega} \left( |u(x, t)|^2 + |\partial_t^2 u(x, t)|^2 \right) dx dt &\leq c(R_0 + R_2) , \\ \int_{\mathbb{R}} \|\partial_t(a_o u)(\cdot, t)\|_{L^2(\Omega)} dt &\leq \int_{\mathbb{R}} \left[ \int_{\Omega} |\partial_t a_o u(x, t)|^2 dx \right]^{1/2} dt + \int_{\mathbb{R}} \left[ \int_{\Omega} |a_o \partial_t u(x, t)|^2 dx \right]^{1/2} dt \\ &\leq \int_{\mathbb{R}} \left| \frac{2t}{c^2} \right| e^{-\frac{1}{c^2} t^2} \left[ \int_{\Omega} |u(x, t)|^2 dx \right]^{1/2} dt + \int_{\mathbb{R}} e^{-\frac{1}{c^2} t^2} \left[ \int_{\Omega} |\partial_t u(x, t)|^2 dx \right]^{1/2} dt \\ &\leq c(R_0 + R_1) , \end{aligned}$$

we conclude that

$$|\mathcal{R}(u)| + |\mathcal{R}(\partial_t^2 u)| \leq c \sqrt{\frac{1}{\lambda}} (R_0 + R_1) (R_0 + R_2) .$$

That completes the proof of (A1).

**Proof of (A2).** We estimate  $\int_{\mathbb{R}^3} \int_{|\tau| < \lambda} \left| \widehat{\varphi f}(\xi, \tau) \right| d\xi d\tau$  where  $f$  solves  $\partial_t^2 f - \Delta f = 0$  in  $\Omega \times \mathbb{R}$ . By Cauchy-Schwarz inequality and Parseval identity,

$$\begin{aligned} \int_{\mathbb{R}^3} \int_{|\tau| < \lambda} \left| \widehat{\varphi f}(\xi, \tau) \right| d\xi d\tau &= \int_{\mathbb{R}^3} \int_{|\tau| < \lambda} \frac{1}{1+|\xi|^2} \left| (1+|\xi|^2) \widehat{\varphi f}(\xi, \tau) \right| d\xi d\tau \\ &\leq \int_{|\tau| < \lambda} \left[ \int_{\mathbb{R}^3} \frac{1}{(1+|\xi|^2)^2} d\xi \right]^{1/2} \left[ \int_{\mathbb{R}^3} \left| \widehat{((1-\Delta)(\varphi f))}(\xi, \tau) \right|^2 d\xi \right]^{1/2} d\tau \\ &\leq \pi^2 \sqrt{\lambda} \left[ \int_{\mathbb{R}^3} \int_{|\tau| < \lambda} \left| \widehat{((1-\Delta)(\varphi f))}(\xi, \tau) \right|^2 d\xi d\tau \right]^{1/2} . \end{aligned}$$

On the other hand, remark that  $\partial_x^j \varphi(x, t) = \frac{1}{h^{j/2}} \phi_j(x) e^{-\frac{1}{c_3 h} |x-x_0|^2} e^{-\frac{1}{c_4} t^2}$  for some  $\phi_j \in C_0^\infty(\Omega)$ . Since  $\Delta(\varphi u) = \varphi \Delta u + \Delta \varphi u + 2\nabla \varphi \nabla u$ , we obtain when  $f = u$ , using Parseval identity and the last remark

$$\begin{aligned} & \int_{\mathbb{R}^3} \int_{|\tau| < \lambda} |\widehat{\varphi u}(\xi, \tau)| d\xi d\tau \\ & \leq c\sqrt{\lambda} \left( \|\varphi \Delta u\|_{L^2(\Omega \times \mathbb{R})} + \|\Delta \varphi u\|_{L^2(\Omega \times \mathbb{R})} + \|\nabla \varphi \nabla u\|_{L^2(\Omega \times \mathbb{R})} + \|\varphi u\|_{L^2(\Omega \times \mathbb{R})} \right) \\ & \leq c\sqrt{\lambda} \left( R_2 + \frac{1}{h} R_0 + \frac{1}{\sqrt{h}} R_1 \right). \end{aligned}$$

Since

$$\begin{aligned} \Delta(\varphi \partial_t^2 u) &= \varphi \partial_t^2 \Delta u + \Delta \varphi \Delta u + 2\nabla \varphi \nabla \partial_t^2 u \\ &= \partial_t^2(\varphi \Delta u) - 2\partial_t(\partial_t \varphi \Delta u) + (\partial_t^2 \varphi + \Delta \varphi) \Delta u + 2\nabla \varphi \nabla \partial_t^2 u \\ &= \partial_t^2(\varphi \Delta u) - 2\partial_t(\partial_t \varphi \Delta u) + (\partial_t^2 \varphi + \Delta \varphi) \Delta u \\ &\quad + 2\partial_t^2(\nabla \varphi \nabla u) - 4\partial_t(\partial_t \nabla \varphi \nabla u) + 2\partial_t^2 \nabla \varphi \nabla u, \end{aligned}$$

we obtain when  $f = \partial_t^2 u$ , using the fact that  $|\tau| < \lambda$ , Parseval identity and the above remark,

$$\begin{aligned} & \int_{\mathbb{R}^3} \int_{|\tau| < \lambda} |\widehat{\varphi \partial_t^2 u}(\xi, \tau)| d\xi d\tau \\ & \leq c\sqrt{\lambda} \left( \lambda^2 \|\varphi \Delta u\|_{L^2(\Omega \times \mathbb{R})} + \lambda \|\partial_t \varphi \Delta u\|_{L^2(\Omega \times \mathbb{R})} + \|(\partial_t^2 \varphi + \Delta \varphi) \Delta u\|_{L^2(\Omega \times \mathbb{R})} + \|\varphi \Delta u\|_{L^2(\Omega \times \mathbb{R})} \right) \\ & \quad + c\sqrt{\lambda} \left( \lambda^2 \|\nabla \varphi \nabla u\|_{L^2(\Omega \times \mathbb{R})} + \lambda \|\partial_t \nabla \varphi \nabla u\|_{L^2(\Omega \times \mathbb{R})} + \|\partial_t^2 \nabla \varphi \nabla u\|_{L^2(\Omega \times \mathbb{R})} \right) \\ & \leq c\sqrt{\lambda} \left( \left( \lambda^2 + \frac{1}{h} \right) R_2 + \frac{\lambda^2}{\sqrt{h}} R_1 \right). \end{aligned}$$

We conclude that there exists  $c > 0$  such that for any  $h \in (0, 1]$  and  $\lambda \geq 1$ ,

$$\int_{\mathbb{R}^3} \int_{|\tau| < \lambda} |\widehat{\varphi u}(\xi, \tau)| d\xi d\tau + \int_{\mathbb{R}^3} \int_{|\tau| < \lambda} |\widehat{\varphi \partial_t^2 u}(\xi, \tau)| d\xi d\tau \leq c\sqrt{\lambda} \left( \left( \lambda^2 + \frac{1}{h} \right) R_2 + \frac{\lambda^2}{\sqrt{h}} R_1 + \frac{1}{h} R_0 \right).$$

That completes the proof of (A2).

## Appendix B

The goal of this Appendix B is to prove the two following inequalities.

**Lemma B .-** For any  $x, y, C, R \in \mathbb{R}$ , any  $z \in \mathbb{C}, \operatorname{Re} z > 0$ ,

$$\begin{aligned} \left| \frac{1}{\sqrt{z}} \sum_{n \in \mathbb{Z}} (-1)^n e^{-\frac{1}{z}(2n+C(-1)^n+R)^2} e^{inx} e^{i(-1)^n y} \right| &\leq \frac{\sqrt{\pi}}{2} + \frac{2}{\sqrt{\operatorname{Re} z}}, \\ \left| \frac{1}{\sqrt{z}} \sum_{n \in \mathbb{Z}} e^{-\frac{1}{z}(2n+C(-1)^n+R)^2} e^{inx} e^{i(-1)^n y} \right| &\leq \frac{\sqrt{\pi}}{2} + \frac{2}{\sqrt{\operatorname{Re} z}}. \end{aligned}$$

Proof .- First we recall the Poisson summation formula. Let  $u \in C^2(\mathbb{R}, \mathbb{C})$  be such that for any  $k \in \{0, 1, 2\}$ , the functions  $x \mapsto (1+x^2) u^{(k)}(x)$  are bounded on  $\mathbb{R}$ . Then for any  $x \in \mathbb{R}$ ,

$$\sum_{n \in \mathbb{Z}} u(x+n) = \sum_{n \in \mathbb{Z}} \widehat{u}(2\pi n) e^{2\pi i n x} \quad \text{where} \quad \widehat{u}(2\pi n) = \int_{\mathbb{R}} u(t) e^{-2\pi i n t} dt.$$

Next, by choosing  $u(x) = v(x) e^{-2\pi i B x}$  for some  $B \in \mathbb{R}$  and  $v \in C^2(\mathbb{R}, \mathbb{C})$  such that for any  $k \in \{0, 1, 2\}$ , the functions  $x \mapsto (1+x^2) v^{(k)}(x)$  are bounded on  $\mathbb{R}$ , we obtain that for any  $x, B \in \mathbb{R}$ ,

$$\sum_{n \in \mathbb{Z}} \widehat{v}(2\pi(n+B)) e^{2\pi i n x} = \sum_{n \in \mathbb{Z}} v(x+n) e^{-2\pi i B(x+n)} \quad \text{where} \quad \widehat{v}(2\pi(n+B)) = \int_{\mathbb{R}} v(t) e^{-2\pi i(n+B)t} dt.$$

Now, for any  $z \in \mathbb{C}$  such that  $\operatorname{Re} z > 0$ , we take  $v(x) = e^{-\frac{z}{2}x^2}$  in order that  $\widehat{v}(2\pi(n+B)) = \frac{\sqrt{2\pi}}{\sqrt{z}} e^{-\frac{1}{2z}(2\pi(n+B))^2}$ . Thus, after simple changes, the following formula holds for any  $x, B \in \mathbb{R}$ , any  $z \in \mathbb{C}, \operatorname{Re} z > 0, a > 0$ ,

$$\frac{1}{\sqrt{z}} \sum_{n \in \mathbb{Z}} e^{-\frac{a}{z}(n+B)^2} e^{i2nx} = \frac{\sqrt{\pi}}{\sqrt{a}} \sum_{n \in \mathbb{Z}} e^{-\frac{z}{a}(x+\pi n)^2} e^{-i2B(x+\pi n)}.$$

Finally, we deduce that for any  $x, y, C, R \in \mathbb{R}$ , any  $z \in \mathbb{C}, \operatorname{Re} z > 0$ ,

$$\begin{aligned} & \left| \frac{1}{\sqrt{z}} \sum_{n \in \mathbb{Z}} (-1)^n e^{-\frac{1}{z}(2n+C(-1)^n+R)^2} e^{inx} e^{i(-1)^n y} \right| \\ &= \left| \frac{1}{\sqrt{z}} \sum_{n \in \mathbb{Z}} e^{-\frac{1}{z}(4n+C+R)^2} e^{i2nx} e^{iy} - \frac{1}{\sqrt{z}} \sum_{n \in \mathbb{Z}} e^{-\frac{1}{z}(4n+2-C+R)^2} e^{i(2n+1)x} e^{-iy} \right| \\ &= \left| e^{iy} \frac{1}{\sqrt{z}} \sum_{n \in \mathbb{Z}} e^{-\frac{4^2}{z} \left(n + \frac{C+R}{4}\right)^2} e^{i2nx} - e^{-iy} e^{ix} \frac{1}{\sqrt{z}} \sum_{n \in \mathbb{Z}} e^{-\frac{4^2}{z} \left(n + \frac{2-C+R}{4}\right)^2} e^{i2nx} \right| \\ &= \left| e^{iy} \frac{\sqrt{\pi}}{4} \sum_{n \in \mathbb{Z}} e^{-\frac{z}{4^2}(x+\pi n)^2} e^{-i\frac{C+R}{2}(x+\pi n)} - e^{-iy} \frac{\sqrt{\pi}}{4} \sum_{n \in \mathbb{Z}} e^{-\frac{z}{4^2}(x+\pi n)^2} e^{-i\frac{2-C+R}{2}(x+\pi n)} \right| \\ &\leq \frac{\sqrt{\pi}}{2} \sum_{n \in \mathbb{Z}} e^{-\frac{\operatorname{Re} z}{4^2} \pi^2 \left(\frac{x}{\pi} + n\right)^2} \leq \frac{\sqrt{\pi}}{2} + \frac{2}{\sqrt{\operatorname{Re} z}}, \end{aligned}$$

and similarly

$$\begin{aligned} & \left| \frac{1}{\sqrt{z}} \sum_{n \in \mathbb{Z}} e^{-\frac{1}{z}(2n+C(-1)^n+R)^2} e^{inx} e^{i(-1)^n y} \right| \\ &= \left| e^{iy} \frac{1}{\sqrt{z}} \sum_{n \in \mathbb{Z}} e^{-\frac{1}{z}(4n+C+R)^2} e^{i2nx} + e^{-iy} e^{ix} \frac{1}{\sqrt{z}} \sum_{n \in \mathbb{Z}} e^{-\frac{1}{z}(4n+2-C+R)^2} e^{i2nx} \right| \\ &\leq \frac{\sqrt{\pi}}{2} + \frac{2}{\sqrt{\operatorname{Re} z}}. \end{aligned}$$

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